A SUPPORT THEOREM FOR HILBERT SCHEMES OF PLANAR CURVES, II

LUCA MIGLIORINI, VIVEK SHENDE, AND FILIPPO VIVIANI

ABSTRACT. We study the cohomology of Jacobians and Hilbert schemes of points on reduced and locally planar curves, which are however allowed to be singular and reducible. We show that the cohomologies of all Hilbert schemes of all subcurves are encoded in the cohomologies of the fine compactified Jacobians of connected subcurves, via the perverse Leray filtration.

CONTENTS

1. Introduction

Given a divisor D on a smooth space C, one can form the line bundle whose sections are those rational functions with poles in D. This construction makes sense in families, and thus defines a map from the space of effective divisors to the space of line bundles

$$\begin{aligned} A : \mathrm{Eff}(\mathrm{C}) &\to \mathrm{Pic}(\mathrm{C}) \\ D &\mapsto \mathcal{O}_C(D) \end{aligned}$$

For singular spaces, various changes must be made. The spaces $\mathrm{Eff}(\mathrm{C})$ and $\mathrm{Pic}(\mathrm{C})$ still make sense, but the map does not. Two problems can already be seen when C is a nodal curve: the sheaf of functions with one pole at the node is not a line bundle, and the sheaf of functions with two poles at the node has degree 3.

When C is proper, reduced, and irreducible, there are natural substitutes [D'S, AIK, AK, AK2]. The space of line bundles is extended to the space $\overline{\mathrm{Pic}}(\mathrm{C})$ of rank one, torsion free sheaves. The space of divisors is replaced by a space $\mathrm{Syst}(\mathrm{C})$ of generalized divisors – rank one, torsion free sheaves equipped with injective sections. There is an evident forgetful map $\mathrm{Syst}(\mathrm{C}) \to \overline{\mathrm{Pic}}(\mathrm{C})$.

When C is proper of dimension 1, these spaces behave in many ways like their classical counterparts. Assuming that C is locally planar, $\overline{\mathrm{Pic}}(\mathrm{C})$ is reduced and irreducible of dimension equal to the arithmetic genus of C, the space $\mathrm{Syst}(\mathrm{C})$ can be identified with the Hilbert scheme, and the above forgetful map is identified with the map sending a subscheme to the dual of its ideal sheaf

$$\begin{aligned} A : \coprod_{n\geqslant 0} \mathrm{C}^{[n]} &\to \overline{\mathrm{Pic}}(\mathrm{C}) \\ D &\mapsto \mathrm{Hom}_C(I_D, \mathcal{O}_C) \end{aligned}$$

Reducibility introduces additional subtleties. Consider the banana curve: two rational components, glued together at two points. The space of line bundles on this curve is $\mathbb{Z} \times \mathbb{Z}$ copies of $\mathbb{G}_m$, where the discrete data gives the degrees of the line bundle on each component. The ability to "take the $(0, 0)$ piece" is lost in the compactification – the torsion free sheaves coming from the nodes serve to glue together the various components of degree $(a, d - a)$.

The problem can be bounded by an appropriate choice of stability condition [Gie, Ses, Sim]. For locally planar curves, it is known that a generic choice leads to a fine moduli space, called a fine compactified Jacobian [Est, MV, MRV1], and moreover, that both its derived category [MRV3] and the topological cohomology (see Theorem 1.8) of the space do not depend on the choice of stability condition. These naturally furnish invariants of the singular curve; we will be interested here in investigating the latter.

We begin with a nodal curve, C. We write $\overline{J}_{\mathrm{C}}$ for the fine compactified Jacobian determined by a fixed but unspecified stability condition. In the introduction, we restrict ourselves to the case where all components of C are rational; for topological purposes, the general case differs from this only by the product of the Jacobians of the components. We write Γ_{C} for the graph whose vertices are the irreducible components of C and whose edges are the nodes joining them.

The space $\overline{J}_{\mathrm{C}}$ is a union of toric varieties glued along toric divisors, by combinatorial rules which can be given in terms of Γ_{C} [OS, Ale, MV]. In particular, the zero dimensional torus orbits are in bijection with spanning trees of Γ. In terms of curves, a spanning tree is a connected partial normalization of arithmetic genus zero. That is:

$$\chi(\overline{J}_{\mathrm{C}}) = \#\{\text{genus zero connected partial normalizations of a nodal curve } \mathrm{C}\}$$

We will write this number as $n_0(\Gamma)$.

A version of the above equality for irreducible curves was used by Yau, Zaslow, and Beauville to count curves on K3 surfaces [YZ, Bea]. It has a certain physical meaning, further elaborated by Gopakumar and Vafa – the right hand side has to do with topological string theory, and the left hand side has to do with BPS D-branes; both are degenerations of some M-theoretic setup, so should be equal [GV]. They also explained that this reasoning explains how to generalize this formula to higher genus. In the case at hand, we promote the right hand side to the number $n_g(\Gamma)$ of genus g connected spanning subgraphs of Γ, or equivalently, the number of genus g connected partial normalizations of the corresponding curve.

Remark 1.1. In graph theory, the spanning subgraphs of Γ are often spoken of in terms of their complements, the *independent* sets of edges. These are the subsets of edges whose images in $H^1(\Gamma,\mathbb{Z}) = \mathrm{Cokernel}(d_\Gamma^* : \mathbb{Z}^{vertices} \to \mathbb{Z}^{edges})$ are linearly independent, so form a matroid. In particular, there is a simplicial complex whose k-simplices are the genus $g(\Gamma)-k$ spanning subgraphs, and thus the $n_{g(\Gamma)-k}$ are counting these simplices.

There are two ways to generalize the left hand side. The first speaks only of the Jacobian, but introduces a filtration on its cohomology. Let $P^iH^*(\overline{J}_{\mathrm{C}},\mathbb{Q})$ be the local perverse Leray filtration, as defined in [MS, MY], on the cohomology of the Jacobian coming from spreading out over any locally versal deformation of C. Let $\mathbb{L}$ be the class of the affine line.

Theorem 1.2. *Let C be a connected nodal curve with rational components, with dual graph Γ. Then we have the following equality in the Grothendieck group of Hodge structures:*

$$\sum_n q^n Gr_P^n H^*(\overline{J}_{\mathrm{C}},\mathbb{Q}) = \sum_h n_h(\Gamma)\cdot (q\mathbb{L})^{g(\Gamma)-h}((1-q)(1-q\mathbb{L}))^h$$

In fact, the original Gopakumar-Vafa prediction spoke only of the specialization $\mathbb{L} = 1$; we are giving a refined version. We will prove a still stronger version of this result, valid in étale cohomology for curves over a finite field. In particular, the Galois group naturally acts on the left hand side, and we will give a Galois-equivariant version of the right hand side. The result as stated follows from Corollary 3.15 combined with Theorem 1.8.

The second generalization of $\chi(\overline{J}_{\mathrm{C}})$ introduces new spaces instead of a cohomological filtration. In general, these spaces should be the $\mathrm{Syst}(\mathrm{C})$ above, or as Pandharipande and Thomas call them, $\mathrm{Pairs}(\mathrm{C})$ [PT]. When C is Gorenstein, and in particular in the locally planar case to which we confine ourselves here, these are isomorphic to the Hilbert schemes. Unlike the Jacobians, the enumerative information contained in these spaces is most naturally related to counting disconnected curves; the two are conjecturally related by an exponential. The pairs spaces were introduced to study enumerative geometry on 3-folds [PT, PT3]; but more relevant to our present work on locally planar curves are their uses in studying curves on surfaces [Sh, KST, KT, KS, GS, GS2], knot invariants [ObS, ORS, GORS, DSV, DHS, Mau], and the geometry of the Hitchin system [CDP].

We introduce some notation. Form the group ring $\mathbb{Z}[[\mathbb{Z}^{vertices}]]$, i.e. the power series ring $\mathbb{Z}[[Q^{v_1}, Q^{v_2},\ldots]]$ on the vertices of the graph. This is where curve counting really happens, but as we count only reduced curves, we pass to the quotient by the ideal $(Q^{2v_1}, Q^{2v_2},\ldots)$. On this quotient ring, we define an exponential

$$\mathbb{E}xp : (Q^{v_1}, Q^{v_2},\ldots)/(Q^{2v_1}, Q^{2v_2},\ldots) \;\;\to\;\; \mathbb{Z}[[Q^{v_1}, Q^{v_2},\ldots]]/(Q^{2v_1}, Q^{2v_2},\ldots)$$

by sending $\mathbb{E}xp(Q^v) = 1 + Q^v$, and requiring that sums go to products.

For any subgraph $\Gamma' < \Gamma$, let $Q^{\Gamma'} := \prod_{v\in\Gamma'} Q^v$. The Hilbert scheme version of the formula is:

Theorem 1.3. *Let C be a connected nodal curve with rational components, with dual graph Γ. Then we have the following equality in the Grothendieck group of Hodge structures:*

$$\sum_{\Gamma'<\Gamma} Q^{\Gamma'}(q\mathbb{L})^{1-g(\Gamma')}\sum_{n=0}^{\infty} q^n H^*(C_{\Gamma'}^{[n]},\mathbb{Q}) = \mathbb{E}xp\left(\sum_{\Gamma'<\Gamma} Q^{\Gamma'}\sum_h n_h(\Gamma')\cdot\left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^{1-h}\right)$$

Recall that, by definition, $n_h(\Gamma')$ vanishes when Γ' is disconnected.

Since the left hand side is computing cohomology of Hilbert schemes and the right hand side is counting maps to the curve, this result is a sort of local motivic MNOP formula for maps to reduced nodal curves [MNOP].[1] We will again prove a Galois-equivariant version over finite fields. The result as stated can be deduced by combining Theorem 1.2 with Corollary 1.18 of Theorem 1.16.[2]

Remark 1.4. We do not know a formula for the Betti numbers of $\overline{J}_{\mathrm{C}}$. Finding such is nontrivial: while the space is built of toric varieties and carries the action of a torus with finitely many fixed points, the cohomology is not equivariantly formal – in particular, there is cohomology in odd degrees.

We turn now to the more general setting of reduced planar curves. Here, the $n_h(\mathrm{C})$ are more mysterious. The closest statement we know to a combinatorial interpretation operates only at the level of Euler characteristics, and asserts that $\chi(n_h(\mathrm{C}))$ is multiplicity of the loci of genus h in a versal deformation of C [Sh]. A conjectural description of the refined invariants in terms of a real structure on the curve can be found in [GS], where we also gave formulas in the case where C is a curve with an ADE singularity [GS]. From these it can be seen that $n_h(\mathrm{C})$ is a nontrivial Hodge structure, although we know of no example in which it is not a polynomial in $\mathbb{L}$.

Nonetheless, we can at least ask for a relation between the analogues of the left hand sides of Theorems 1.2 and 1.3.

In the case of a single smooth curve C, the cohomologies of the Hilbert schemes $C^{[n]}$ – in this case, just the symmetric products – and the Jacobian $J(C)$ can both be built from $H^1(C,\mathbb{Q})$. Explicitly:

$$\bigoplus_{n=0}^{\infty} q^n\cdot H^*(C^{[n]},\mathbb{Q}) = Sym^*(qH^*(C,\mathbb{Q}))\cong\frac{\bigoplus q^i\cdot\wedge^i H^1(C,\mathbb{Q})[-i]}{(1-q)(1-q\mathbb{L})} = \frac{\bigoplus q^i H^i(J(C),\mathbb{Q})[-i]}{(1-q)(1-q\mathbb{L})}$$

Here, $\mathbb{L}:=[-2](-1)$, and as usual the symmetric powers of a vector space in odd cohomological degree really mean its exterior powers.

The formula works in families: given a smooth family of curves $\pi_{sm}:\mathcal{C}\to B_{sm}$,

$$\bigoplus_{n=0}^{\infty} q^n\cdot R\pi_{sm*}^{[n]}\mathbb{Q} = Sym^*(q\cdot R\pi_{sm*}\mathbb{Q})\cong\frac{\bigoplus q^i\cdot\wedge^i R^1\pi_{sm*}\mathbb{Q}[-i]}{(1-q)(1-q\mathbb{L})} = \frac{\bigoplus q^i\cdot R^i\pi_{sm*}^J\mathbb{Q}[-i]}{(1-q)(1-q\mathbb{L})}$$

Now consider a family $\pi_\heartsuit:\mathcal{C}\to B_\heartsuit$ of reduced, irreducible locally planar curves. We can form the relative Hilbert scheme $\pi_\heartsuit^{[n]}:\mathcal{C}^{[n]}\to B_\heartsuit$, and the relative compactified Jacobian $\pi_\heartsuit^J:\overline{J}(\mathcal{C})\to$

[1]The usual context of such formulas is the counting of curves in 3-dimensional Calabi-Yau varieties, in which case the stable pairs moduli space is generally singular and the Euler characteristics and cohomlogies discussed here must be corrected by the Behrend function [Beh] or its cohomological upgrade. The formulas here will apply to this 3-fold setting only in the case that the moduli space is smooth, which however can happen, e.g. when the 3-fold contains an isolated surface.

[2]This is technically correct, but it is conceptually less circular to say that the result follows from Proposition 3.5 plus the combinatorics in the proof of Theorem 1.16.

$B_\heartsuit$. If all the relative Hilbert schemes have nonsingular total space, then the same is true for the relative compactified Jacobian. In [MY, MS], the families of cohomologies $R\pi^{[n]}_{\heartsuit *}\mathbb{Q}$ and $R\pi^{J}_{\heartsuit *}\mathbb{Q}$ were shown to enjoy the following relation:

$$\bigoplus_{n=0}^{\infty} q^n \cdot R\pi^{[n]}_{\heartsuit *}\mathbb{Q} \cong \frac{\bigoplus q^i \cdot IC(\wedge^i R^1\pi_{sm*}\mathbb{Q})[-i]}{(1-q)(1-q\mathbb{L})} = \frac{\bigoplus q^i \cdot {}^pR^i\pi^J_{\heartsuit *}\mathbb{Q}[-i]}{(1-q)(1-q\mathbb{L})}$$

Here, IC denotes the intersection cohomology sheaf extending the given local system and ${}^pR^i f_* := {}^p\mathcal{H}^i(Rf_*)$ means the i'th perverse cohomology sheaf of the derived pushforward. We take the convention that intersection cohomology complexes 'begin in degree zero', so K is perverse in our sense if $K[\dim B]$ is perverse in the sense of [BBD], see §2.1.8 .

We recall a few ideas from the proof. It follows from the 'decomposition theorem' of [BBD] that the middle term above is a direct summand both on the right and the left, and any other summands must have positive codimensional support, so it remains only to show that there are no such summands. On the RHS, hence on the LHS for $n \gg 0$ via the Abel-Jacobi map, this is a consequence of the 'support theorem' of [Ngô]. In [MY], this is bootstrapped to an argument for the LHS by constructing correspondences between the Hilbert schemes. In [MS], we take a different approach, suitable for both the LHS and RHS, to reduce checking to the nodal locus, where it may be done explicitly. We have since abstracted this method into the theory of higher discriminants [MS2]. Yet another approach to similar results can be found in [Ren].

Our present goal is to establish such a comparison over the locus of reduced curves – i.e., to treat the reduced but not necessarily irreducible case. As we already mentioned, there are already subtleties in the definition of the compactified Jacobian, but so long as the curves lie in a fixed surface or fixed family of surfaces or we are working in a neighborhood of some fixed curve, we can choose compatible stability conditions over the whole base and consider the relative fine compactified Jacobian $\pi^J : \overline{\mathcal{J}} \to B$. Second, due to the above stability issues, there is no Abel-Jacobi map directly relating the Hilbert schemes and the Jacobians.

Third, it is no longer true in general that smoothness of $\overline{\mathcal{J}}$ guarantees the absence of summands of $R\pi^J_*\mathbb{C}$ with positive codimensional supports.

Example 1.5. Consider a one-parameter family of elliptic curves degenerating to a cycle of $\geqslant 2$ $\mathbb{P}^1$'s. This family is its own relative fine compactified Jacobian [MRV1, Prop. 7.3], but evidently $R\pi^J_*\mathbb{Q}$ has a summand supported at the special point to account for its extra H^2.

Nonetheless, over sufficiently big families, this phenomenon does not occur.

Definition 1.6. We say $\pi : \mathcal{C} \to B$ is *H-smooth* if all relative Hilbert schemes $\mathcal{C}^{[n]}$ have smooth total space. Note this includes $\mathcal{C}^{[0]} = B$.

Example 1.7. Over any field, a versal family for a reduced curve with planar singularities is H-smooth, see e.g. [Sh, Prop. 17].

Theorem 1.8. *Let $\pi : \mathcal{C} \to B$ be H-smooth. Then no summand of $R\pi^J_*\overline{\mathbb{Q}}_\ell$ has positive codimensional support. Thus, ${}^pR^i\pi^J_*\overline{\mathbb{Q}}_\ell \cong IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_\ell)$, and the stalk at $[C]$ of ${}^pR^i\pi^J_*\overline{\mathbb{Q}}_\ell$ does not depend on the choice of the H-smooth family $\mathcal{C}$. Over a field of characteristic zero, $\overline{\mathbb{Q}}_\ell$ can be replaced by $\mathbb{Q}$.*

In some cases, this follows from the work of Chaudouard and Laumon [CL]. To prove the result, we use the method of higher discriminants [MS2], plus the following smoothness criterion, to reduce the result to the case of irreducible curves, where it is known [Ngô, MS].

Theorem 1.9. *Let $\pi : (\mathcal{C}, \mathrm{C}) \to (S, 0)$ be a projective flat family of connected locally planar curves, with distinguished special fibre $\mathrm{C} \to 0$. Let $d\sigma : T_o S \to T\,\mathrm{Def}^{\mathrm{loc}}(\mathrm{C})$ be the induced map to the first-order deformation of the singularities of C. Let $\gamma(\mathrm{C})$ be the number of connected components of C, and $\delta(\mathrm{C})$ its cogenus.*

If $\mathrm{Im}(\mathrm{d}\sigma)$ is a generic subspace of $\mathbb{V}$ of dimension at least $\delta(\mathrm{C}) + 1 - \gamma(\mathrm{C})$, then the relative compactified Jacobian $\overline{J}_{\mathcal{C}}$ is regular along the special fibre $\overline{J}_{\mathrm{C}}$.

A more precise version of Theorem 1.9 can be found as Theorem 4.11.

Even for H-smooth families – even for versal families – there are many summands of $R\pi_*^{[n]}\mathbb{C}$ which are supported in positive codimension. In fact, at a reducible curve $[\mathrm{C}] \in B$, there is such a summand for every splitting of C into connected subcurves.

Nonetheless, we will establish various analogues of the main result of [MY, MS], both at a single curve, and globally for what we call *independently broken* families.

Definition 1.10. Let V be a finite set. An independently broken family of reduced planar curves indexed by V is:

- For each $S \subset V$, a space B_S, and a flat family of reduced planar curves $\mathcal{C}_S \to B_S$
- For each disjoint union $S' \coprod S'' = S$, an injective map $B_{S'} \times B_{S''} \to B_S$
- Fibrewise partial normalizations $\mathcal{C}_{S'} \times B_{S''} \coprod B_{S'} \times \mathcal{C}_{S''} \to \mathcal{C}_S|_{B_{S'} \times B_{S''}}$
- Such that every decomposition into two (not necessarily irreducible) components of each curve is uniquely realized by some such partial normalization
- And so that the evident compatibilities for iterated disjoint unions are satisfied.

Remark 1.11. What is being ensured by the above definition is that irreducible components can be varied independently, both in the local sense that they can be deformed independently, and in the global sense that there can be no possible monodromy of their names.

Definition 1.12. An independently broken family of reduced planar curves indexed by V is H-smooth if, for each $S \subset V$, the family of reduced planar curves $\mathcal{C}_S \to B_S$ is H-smooth.

Example 1.13. (Versal families.) Let C be a reduced planar curve. Let V be the set of its irreducible components, and for any $S \subset V$, let C_S be the sub-curve formed by just the corresponding irreducible components. Then a miniversal family[3] $\mathcal{C} \to \mathrm{Def}(\mathrm{C})$ can be extended to an independently broken family by adding the miniversal deformations of subcurves $\mathcal{C}_S \to B_S := \mathrm{Def}(\mathrm{C}_S)$. This family is H-smooth (see Theorem 4.15).

Example 1.14. (Parabolic Hitchin systems.) Let X be a smooth curve, choose a point $x \in X$, a number $d > 0$, and consider the surface Y which is the total space of $\omega_X(d \cdot x)$. We write $\mathcal{O}_Y(n \cdot X)$ for the linear system of compact curves in Y which are linearly equivalent to n copies of the zero section. Choose a general fibre of $Y \to X$ and choose n general points, $p_1, \ldots, p_n$ on this fibre.

Let $V = \{p_1, \ldots, p_n\}$. For a given subset $S \subset V$, we take $\mathcal{C}_S \to B_S$ to be the family of curves in $\mathcal{O}_Y(|S| \cdot X)$ which pass through the points in S. Note all curves in this family are reduced because the general points p_i are in particular distinct. The structure maps are given by taking unions of curves inside Y. This is an independently broken family of curves.

[3]As versal deformations are not unique, there is some ambiguity amongst the following choices, upon which nothing depends. Here and throughout the paper, it suffices to take families which induce versal deformations of the singularities of C. We always understand the base of a versal deformation as an étale germ around a central point $[\mathrm{C}]$, which is to say, we allow ourselves to pass to a smaller étale neighborhood of this point in the base at will. This will be necessary to ensure that there is no monodromy of the irreducible components of C in the equigeneric stratum; see Lemma 2.3.

Example 1.15. (Nonexample: conics in $\mathbb{P}^2$) Let $U \subset \mathbb{P}^5$ be the Zariski open subset parameterizing reduced plane conics, and $S \subset U$ the locus of reducible conics. The space S is the quotient by an involution of the complement of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$; in particular, $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$. This loop corresponds to the monodromy of a pair of lines around a double line, in which the two components are exchanged. Thus the tautological family over U is not independently broken. However, it does give a versal deformation of singularities, hence is in particular H-smooth. Notice that U is simply connected since $\mathbb{P}^5 \backslash U$ has codimension three.

In the context of an independently broken family, we have an 'exponential map' which acts on the category of sheaves on $\coprod_S B_S$ by

$$\mathbb{E}xp(\mathcal{F})|_{B_S} := \bigoplus_{S=\coprod S_i} \boxtimes (\mathcal{F}|_{B_{S_i}}) \tag{1}$$

Here, we have identified $\prod B_{S_i}$ with a subscheme of B_S via the structure maps.

Theorem 1.16. *Let $\mathcal{C}_S \to B_S$ be an H-smooth independently broken family of locally planar curves over an algebraically closed field, admitting relative fine compactified Jacobians $\overline{\mathcal{J}}_S \to B_S$. By definition the relative fine compactified Jacobian of a disconnected curve is empty. Let g denote the locally constant function giving the arithmetic genus of the curves being parameterized.*

Then there are isomorphisms in $D_c^b(\coprod B_S)[[q]]$:

$$\begin{aligned} (q\mathbb{L})^{1-g} \bigoplus_{n=0}^{\infty} q^n R\pi_*^{[n]} \overline{\mathbb{Q}}_l &\cong \mathbb{E}xp\left((q\mathbb{L})^{1-g} \cdot \frac{\bigoplus q^i \cdot IC(\bigwedge^i R^1 \pi_{sm*} \overline{\mathbb{Q}}_l)[-i]}{(1-q)(1-q\mathbb{L})} \right) \\ &\cong \mathbb{E}xp\left((q\mathbb{L})^{1-g} \cdot \frac{\bigoplus q^i \cdot {}^p R^i \pi_*^J \overline{\mathbb{Q}}_l[-i]}{(1-q)(1-q\mathbb{L})} \right). \end{aligned}$$

Example 1.17. Let C be the union of pair of lines, $\mathrm{C}_1, \mathrm{C}_2$ which meet once and transversely. A representative for $\mathrm{Def}(\mathrm{C})$ is given by taking the compactification of the map $(x, y) \mapsto xy$; in any case we denote this deformation $\mathcal{C}_{\{1,2\}} \to B_{\{1,2\}}$. We complete this to an independently broken family by taking $\mathrm{C}_i = \mathcal{C}_{\{i\}} \to B_{\{i\}} = \text{point}$ (and $\varnothing = \mathcal{C}_\varnothing \to B_\varnothing = \varnothing$).

We want to compute directly the LHS and RHS of Theorem 1.16. We will just study the stalks at the point $[\mathrm{C}]$. One has, e.g. from [Ran],

$$[\mathrm{C}^{[n]}] = [(\mathbb{P}^1)^{[n]}] + [(\mathbb{P}^1 \coprod \mathbb{P}^1)^{[n-1]})] \cdot \mathbb{L}$$

hence the LHS is given by:

$$\begin{aligned} (q\mathbb{L})^{1-g} \bigoplus_{n=0}^{\infty} q^n R\pi_*^{[n]} \overline{\mathbb{Q}}_l|_{[\mathrm{C}]} &= (q\mathbb{L})^1 \sum_{n=0}^{\infty} q^n \left([\mathbb{P}^n] + \mathbb{L} \cdot \sum_{j=0}^{n-1} [\mathbb{P}^j][\mathbb{P}^{n-1-j}] \right) \\ &= \frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})} + \left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})} \right)^2 \end{aligned}$$

On the RHS, we must compute the exponential. This amounts to summing over decompositions of the curve C; here there are just two, $\mathrm{C} = \mathrm{C}$ and $\mathrm{C} = \mathrm{C}_1 \cup \mathrm{C}_2$. All genera are zero and (hence) all fine compactified Jacobians are just points. Thus, the contribution of $\mathrm{C} = \mathrm{C}$ is $\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}$, and the contribution of $\mathrm{C} = \mathrm{C}_1 \cup \mathrm{C}_2$ is $\left(\frac{q\mathbb{L}}{(1-q)(1-q\mathbb{L})}\right)^2$.

We briefly indicate the shape of the proof. Note first that the left hand side is independent of the family, and the right hand side is, by 1.8, independent of the choice of H-smooth family. To check this equality, in fact we will observe that both sides are pure complexes, base change to a finite field, show they have the same Frobenius traces at closed points, and then conclude by Cebotarev that they are equal. (This is why we require $\overline{\mathbb{Q}}_\ell$ coefficients.) This argument is pointwise on the base, so we may pass now to the versal family. Here, we use our knowledge of the maps π^J and $\pi^{[n]}$ [FGvS, Sh, MRV2] to see that at-worst-nodal curves are dense in what are called in [MS2] the *higher discriminants of these maps*. Thus by [MS2] we are reduced to proving the statement for families of at-worst-nodal curves, where we establish the result by explicitly computing Frobenius traces of stalks. Determining these traces for the IC sheaves $IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_l)$ is the essential computation, which we perform in Section 3.4 using the Cattani-Kaplan-Schmidt complex [CKS].

Theorem 1.16 has the following local and global corollaries, by taking stalk or cohomology.

Corollary 1.18. *Let* C *be a reduced planar curve. We write* $\mathrm{C}' < \mathrm{C}$ *to indicate a subcurve. There is an isomorphism*

$$\bigoplus_{\mathrm{C}'<\mathrm{C}} Q^{\mathrm{C}'}(q\mathbb{L})^{1-g(\mathrm{C}')}\bigoplus_{n=0}^{\infty} q^n H^*((\mathrm{C}')^{[n]};\overline{\mathbb{Q}}_\ell) = \mathbb{E}xp\left(\sum_{\mathrm{C}'<\mathrm{C}} \frac{Q^{\mathrm{C}'}(q\mathbb{L})^{1-g(\mathrm{C}')}}{(1-q)(1-q\mathbb{L})}\bigoplus_i q^i Gr_P^i H^*(\overline{J}_{\mathrm{C}'};\overline{\mathbb{Q}}_\ell)\right)$$

Here, $Gr^i_P H^*(\overline{J}_{\mathrm{C}'};\overline{\mathbb{Q}}_\ell)$ *is by definition* ${}^pR^i\pi^J_*\overline{\mathbb{Q}}_\ell[-i]|_{[\mathrm{C}']}$ *with respect to any H-smooth family containing* C' *and* $\overline{J}_{\mathrm{C}'}$ *is any fine compactified Jacobian of* C' *(with the convention that* $\overline{J}_{\mathrm{C}'}$ *is the empty set for disconnected* C'*).*

Corollary 1.19. *Let* V *be a finite set and* $\{\mathcal{C}_S \to B_S\}_{V\subset S}$ *be an independently broken family with quasiprojective bases* B_S*, admitting relative fine compactified Jacobians* $\pi^J_S : \overline{\mathcal{J}}_S \to B_S$*. Then there is an isomorphism*

$$\bigoplus_{S'\subset S} Q^{S'}(q\mathbb{L})^{1-g(\mathcal{C}_{S'})}\bigoplus_{n=0}^{\infty} q^n H^*(\mathcal{C}_{S'}^{[n]};\overline{\mathbb{Q}}_\ell) = \mathbb{E}xp\left(\sum_{S'<S} \frac{Q^{S'}(q\mathbb{L})^{1-g(\mathcal{C}_{S'})}}{(1-q)(1-q\mathbb{L})}\bigoplus_i q^i Gr_P^i H^*(\overline{\mathcal{J}}_{S'};\overline{\mathbb{Q}}_\ell)\right)$$

Here, $Gr^i_P H^*(\overline{\mathcal{J}}_{S'};\overline{\mathbb{Q}}_\ell)$ *is the associated graded of the global perverse Leray filtration coming from the map* $\overline{\mathcal{J}}_{S'} \to B_{S'}$*.*

The point of these results is that the perverse filtration appears prominently in recent studies of the cohomology of the Hitchin system [dCHM, CDP] and its fibres [GORS, OY], but is difficult to compute directly. On the other hand, the cohomology of the Hilbert schemes is more directly accessible, and the theorem explains how to recover the associated graded pieces of the perverse filtration on the Jacobian from the collection of all cohomologies of the Hilbert schemes.

This sort of relation was in a certain sense predicted in the physics literature [GV, KKV, HST, CDP] as a relation between refined Gopakumar-Vafa invariants (here, the Jacobians) and the refined Donaldson-Thomas invariants (here, the Hilbert schemes).

Acknowledgements. We thank Mark A. de Cataldo and Tamás Hausel for helpful discussions. L.M. is partially supported by PRIN project 2012 "Spazi di moduli e teoria di Lie". During the preparation of this paper L.M. was a member of the School of Mathematics of the Institute for Advanced Study in Princeton, partially funded by the Giorgio and Elena Petronio fellowship. V. S. is supported by the NSF grant DMS-1406871, and by a Sloan fellowship. F.V. is partially supported by the MIUR project "Spazi di moduli e applicazioni" (FIRB 2012).

2. Background

2.1. Notation.

2.1.1. A **curve** is a *reduced* (but not necessarily geometrically irreducible) scheme of pure dimension 1 over a *perfect* field k. In practice we take k to be the complex numbers ($\mathbb{C}$), a finite field ($\mathbb{F}_\pi$), or the algebraic closure of a finite field ($\overline{\mathbb{F}}_\pi$).

Unless otherwise specified, a curve is meant to be projective.

2.1.2. Given a curve C, we denote by C_{sm} the smooth locus of C, by $\mathrm{C}_{\mathrm{sing}}$ its singular locus and by $\nu : \mathrm{C}^\nu \to \mathrm{C}$ the normalization morphism.

2.1.3. We employ the following names and notation for numerical invariants of a curve C:

name	notation	formula
number of irreducible components	$\gamma(\mathrm{C})$	
arithmetic genus	$g(\mathrm{C})$	$1 - \chi(\mathcal{O}_\mathrm{C})$
geometric genus	$g(\mathrm{C}^\nu)$	
cogenus, or total delta invariant	$\delta(\mathrm{C})$	$g(\mathrm{C}) - g(\mathrm{C}^\nu)$
abelian rank	$g^\nu(\mathrm{C})$	$g(\mathrm{C}^\nu) - 1 + \gamma(C)$
affine rank	$\delta^a(\mathrm{C})$	$\delta(C) + 1 - \gamma(C) = g(\mathrm{C}) - g^\nu(\mathrm{C})$

Recall that the cogenus is equal to the sum of the local delta invariants of the singularities:

$$\delta(\mathrm{C}) := \sum_{q \in \mathrm{C}_{\mathrm{sing}}} [k(q) : k] \cdot \delta(\mathrm{C}, q) = \sum_{q \in \mathrm{C}_{\mathrm{sing}}} [k(q) : k] \cdot \mathrm{length}(\nu_* \mathcal{O}_{\mathrm{C}^\nu} / \mathcal{O}_\mathrm{C})_q.$$

The terminology "affine rank" and "abelian rank" will be explained in 2.1.6. Note the abelian rank is also equal to the sum of the genera of the connected components of the normalization.

The cogenus $\delta(\mathrm{C})$ and the affine rank $\delta^a(\mathrm{C})$ are upper semicontinuous in flat families of curves (see [DH, Prop. 2.4] or [GLS, Chap. II, Thm. 2.54] in characteristic zero and [Lau2, Prop. A.2.1] and [MRV2, Lem. 3.2] in arbitrary characteristic). Equivalently, the geometric genus and the abelian rank are lower semicontinuous.

2.1.4. A curve C is **locally planar at** $p \in \mathrm{C}$ if the completion $\widehat{\mathcal{O}}_{\mathrm{C},p}$ of the local ring of C at p has embedded dimension two, i.e., $\widehat{\mathcal{O}}_{\mathrm{C},p} \cong k[[x,y]]/(f)$, for some reduced $f = f(x,y) \in k[[x,y]]$.

A curve C is locally planar if it is locally planar at every $p \in \mathrm{C}$. Being locally a divisor in a smooth space, a locally planar curve is Gorenstein, i.e. the dualizing sheaf ω_C is a line bundle.

2.1.5. A **subcurve** D of a curve C is a reduced subscheme of pure dimension 1. We say that a sub-curve $D \subseteq \mathrm{C}$ is non-trivial if $D \neq \emptyset, \mathrm{C}$.

2.1.6. Given a curve C, the **generalized Jacobian** of C, denoted by J_C or by $\mathrm{Pic}^{\underline{0}}(\mathrm{C})$, is the connected component of the Picard scheme $\mathrm{Pic}(\mathrm{C})$ of C containing the identity, see [BLR, §8.2, Thm. 3] and references therein for existence theorems. The generalized Jacobian of C is a connected commutative smooth algebraic group of dimension equal to $h^1(\mathrm{C}, \mathcal{O}_\mathrm{C})$. Under mild hypotheses such as existence of a rational k-point, or triviality of the Brauer group of k, certainly met in the cases $k = \mathbb{F}_\pi, \overline{\mathbb{F}}_\pi, \mathbb{C}$, its group of k'-valued points, for k' a finite extension of k, parameterizes line bundles on C of multidegree $\underline{0}$ (i.e. having degree 0 on each irreducible component of C) with the multiplication given by the tensor product.

From the exact sequence of sheaves on C

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \nu_*\mathbb{G}_m \longrightarrow \nu_*\mathbb{G}_m/\mathbb{G}_m \longrightarrow 1$$

where $\nu : \mathrm{C}^\nu \to \mathrm{C}$ the normalization morphism, it follows easily that the generalized Jacobian J_{C} is an extension of an abelian variety of dimension $g^\nu(\mathrm{C})$ (namely the Jacobian of the normalization C^ν) by an affine algebraic group of dimension equal to $\delta^a(\mathrm{C})$.

2.1.7. We use $\mathbb{L}$ to mean "whatever incarnation of the Lefschetz motive is appropriate". That is, if we are discussing ungraded vector spaces in the presence of weights, e.g. the K group of mixed Hodge structures or of continuous $\hat{\mathbb{Z}}$ representations over $\overline{\mathbb{Q}}_\ell$, we mean a one dimensional vector space twisted by (-1). If we are working with graded vector spaces, i.e. in the derived category of the above rather than the K group, we mean a one dimensional vector space, twisted by (-1), and placed in cohomological degree 2, e.g. $\mathbb{L} = \overline{\mathbb{Q}}_\ell(-1)[-2]$. In the Grothendieck ring of varieties $\mathbb{L}$ is the class of the affine line.

2.1.8. *Intersection cohomology and perverse sheaves.* In this paper we use the convention according to which the intersection cohomology complex $IC(L)$ of a local system L on a dense open set Z^0 of a nonsingular variety Z restricts to L, as opposed to $L[\dim Z]$. In our convention we say K is perverse on Z if and only if $K[\dim Z]$ is perverse in the sense of [BBD]. Thus, given a local system L' on a locally closed $Z' \subset Z$, the complex $IC(L')[-\mathrm{codim} Z']$ is perverse.

2.2. **Deformation theory of locally planar curves.** We recall facts about the deformation theory of of locally planar curves and their simultaneous desingularization. These facts are well known over the complex numbers; original proofs can be found in the papers [Tes, DH] and a textbook treatment in [GLS]. They have also been partially extended to positive characteristic in [Lau2], [MY], [MRV2]. For maximal accessibility we give in the footnotes precise references to the book of Sernesi [Ser] for some of the standard deformation theoretic facts we use. Deformation theory operates with formal schemes, and we use Spf to denote the formal spectrum.

Let $\mathrm{Def}_{\mathrm{C}}$ be the deformation functor of a (reduced and projective) curve C.[4] For $p \in \mathrm{C}_{\mathrm{sing}}$, we denote by $\mathrm{Def}_{\mathrm{C},p}$ the deformation functor of the complete local k-algebra $\widehat{\mathcal{O}}_{\mathrm{C},p}$. [5] There is a natural transformation of functors

$$\mathrm{Def}_{\mathrm{C}} \to \mathrm{Def}_{\mathrm{C}}^{\mathrm{loc}} := \prod_{p \in \mathrm{C}_{\mathrm{sing}}} \mathrm{Def}_{\mathrm{C},p}\,. \tag{2}$$

If C has locally planar singularities (or, more generally, locally complete intersection singularities), the functors $\mathrm{Def}_{\mathrm{C}}$ and $\mathrm{Def}_{\mathrm{C}}^{\mathrm{loc}}$ are smooth [6] and the morphism (2) is smooth.[7]

The functor $\mathrm{Def}_{\mathrm{C}}$ admits a semiuniversal (or miniversal) formal couple[8] $(R_{\mathrm{C}}, \overline{\mathcal{C}})$, where R_{C} is a Noetherian complete local k-algebra with maximal ideal $\mathfrak{m}_{\mathrm{C}}$ and residue field k and

$$\overline{\mathcal{C}} \in \widehat{\mathrm{Def}_{\mathrm{C}}}(R_{\mathrm{C}}) := \varprojlim \mathrm{Def}_{\mathrm{C}}\left(\frac{R_{\mathrm{C}}}{\mathfrak{m}_{\mathrm{C}}^n}\right)$$

is a formal deformation of C over R_{C}. That is, the morphism of functors

$$h_{R_{\mathrm{C}}} := \mathrm{Hom}(R_{\mathrm{C}}, -) \longrightarrow \mathrm{Def}_{\mathrm{C}} \tag{3}$$

[4][Ser, Sec. 2.4.1]
[5][Ser, Sec. 1.2.2]
[6][Ser, Cor. 3.1.13(ii) and Ex. 2.4.9]
[7][Ser, Prop. 2.3.6]
[8][Ser, Cor. 2.4.2]

determined by $\overline{\mathcal{C}}$ is smooth and induces an isomorphism of tangent spaces[9]

$$TR_{\mathrm{C}} := (\mathfrak{m}_{\mathrm{C}}/\mathfrak{m}_{\mathrm{C}}^2)^\vee \xrightarrow{\cong} T\,\mathrm{Def}_{\mathrm{C}}$$

The formal couple $(R_{\mathrm{C}}, \overline{\mathcal{C}})$ can also be viewed as a flat morphism of formal schemes[10]

$$\overline{\pi} : \overline{\mathcal{C}} \to \mathrm{Spf}\ \mathrm{R}_{\mathrm{C}}, \tag{4}$$

such that the fiber over $[\mathfrak{m}_{\mathrm{C}}] \in \mathrm{Spf}\ \mathrm{R}_{\mathrm{C}}$ is isomorphic to C.

Since C has locally planar singularities, and hence $\mathrm{Def}_{\mathrm{C}}$ is a smooth functor, the semiuniversal formal couple $(R_{\mathrm{C}}, \overline{\mathcal{C}})$ is unique[11] and R_C is formally smooth: $R_C \cong k[[T_1, \ldots, T_N]]$ for some N.

Since C is projective and $H^2(\mathrm{C}, \mathcal{O}_{\mathrm{C}}) = 0$, Grothendieck's existence theorem[12] gives that the formal deformation (4) is *effective*, i.e. there exists a deformation $\pi : \mathcal{C} \to \mathrm{Def}(\mathrm{C})$ of C over $\mathrm{Def}(\mathrm{C}) := \mathrm{Spec}\,\mathrm{R}_{\mathrm{C}}$ whose completion along $\mathrm{C} = \pi^{-1}([\mathfrak{m}_{\mathrm{C}}])$ is isomorphic to (4). In other words, we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{C} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{C}} \arrow[r] \arrow[d, "\overline{\pi}"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "\pi"] \\ \mathrm{Spec}\ \mathrm{k} \cong [\mathfrak{m}_{\mathrm{C}}] \arrow[r, hook] & \mathrm{Spf}\ \mathrm{R}_{\mathrm{C}} \arrow[r] & \mathrm{Spec}\ \mathrm{R}_{\mathrm{C}} = \mathrm{Def}(\mathrm{C}) \end{tikzcd} \tag{5}$$

The deformation π is unique.[13]

The space $\mathrm{Def}(\mathrm{C})$ admits a stratification (called the *equigeneric stratification*) into locally closed subsets according to the cogenus of the geometric fibers of the family π. More precisely, using the notation introduced in 2.1.1, consider the cogenus function

$$\begin{aligned} \delta : \mathrm{Def}(\mathrm{C}) &\longrightarrow \mathbb{N}, \\ t &\mapsto \delta(\mathcal{C}_{\overline{t}}), \end{aligned} \tag{6}$$

where $\mathcal{C}_{\overline{t}} := \pi^{-1}(t) \times_{k(t)} \overline{k(t)}$ is a geometric fiber of π over the point $t \in \mathrm{Def}(\mathrm{C})$.

We call the strata of constant cogenus the *equigeneric strata*, and write

$$\mathrm{Def}(\mathrm{C})^{\delta=d} := \{t \in \mathrm{Def}(\mathrm{C}) : \delta(\mathcal{C}_{\overline{t}}) = d\} \tag{7}$$

$$\mathrm{Def}(\mathrm{C})^{\delta\geqslant d} := \{t \in \mathrm{Def}(\mathrm{C}) : \delta(\mathcal{C}_{\overline{t}}) \geqslant d\} \tag{8}$$

These are empty unless $0 \leqslant d \leqslant \delta(\mathrm{C}) = g(\mathrm{C}) - g(\mathrm{C}^\nu)$, and by upper semicontinuity of δ (see 2.1.3) we have $\mathrm{Def}(\mathrm{C})^{\delta\geqslant d} = \overline{\mathrm{Def}(\mathrm{C})^{\delta=d}}$.

The main properties of the equigeneric strata are contained in the following result, due originally to Teissier and Diaz-Harris if $k = \mathbb{C}$ (see [GLS, Chap. II]), and subsequently extended to fields of big characteristics in [MY, Prop. 3.5] and then to fields of arbitrary characteristics in [MRV2, Thm. 3.3].

Fact 2.1. *Assume that* C *is a (reduced and projective) curve with locally planar singularities. Then, for any* $0 \leqslant d \leqslant \delta(\mathrm{C})$,

(i) The closed subset $\mathrm{Def}(\mathrm{C})^{\delta\geqslant d} \subset \mathrm{Def}(\mathrm{C})$ *has codimension at least* d.
(ii) Each generic point η *of* $\mathrm{Def}(\mathrm{C})^{\delta\geqslant d}$ *is such that* $\mathcal{C}_{\overline{\eta}}$ *is a nodal curve with* d *nodes.*

---

[9] [Ser, Sec. 2.2]
[10] [Ser, p. 77]
[11] [Ser, Prop. 2.2.7]
[12] [Ser, Thm. 2.5.13]
[13] [Ser, Thm. 2.5.11]

On the normalization of each equigeneric stratum of $\mathrm{Def}(\mathrm{C})$, the pull-back of the universal family $\pi : \mathcal{C} \to \mathrm{Def}(\mathrm{C})$ admits a simultaneous normalization. More precisely we have the following result which was originally proved in [Tes, 1.3.2] if $k = \mathbb{C}$ and then extended to arbitrary fields in [Lau2, Prop. A.2.1].

Fact 2.2. *Assume that* C *is a (reduced and projective) curve with locally planar singularities. For any* $0 \leqslant d \leqslant \delta(\mathrm{C})$, *consider the normalization* $\widetilde{\mathrm{Def}(\mathrm{C})^{\delta=d}}$ *of the equigeneric stratum with cogenus* d *and denote by* $\pi^d : \mathcal{C}^{\delta=d} \to \widetilde{\mathrm{Def}(\mathrm{C})^{\delta=d}}$ *the pull-back of the universal family* $\pi : \mathcal{C} \to \mathrm{Def}(\mathrm{C})$. *Then the normalization* $\nu^d : \widetilde{\mathcal{C}^{\delta=d}} \to \mathcal{C}^{\delta=d}$ *is a simultaneous normalization of the family* π^d, *i.e.*

(i) the composition $\nu^d : \widetilde{\mathcal{C}^{\delta=d}} \xrightarrow{\nu^d} \mathcal{C}^{\delta=d} \xrightarrow{\pi^d} \widetilde{\mathrm{Def}(\mathrm{C})^{\delta=d}}$ *is smooth;*
(ii) the morphism ν^d *induces the normalization morphism on each geometric fiber of* π^d.

Using the above Fact, we will prove the following result which is needed later on.

Lemma 2.3. *With the same assumptions as before, consider the equigeneric stratum of maximal cogenus,* $\Delta := \mathrm{Def}(\mathrm{C})^{\delta=\delta(\mathrm{C})}$, *and let* $\mathcal{C}_\Delta \to \Delta$ *be the restriction of the universal family* $\pi : \mathcal{C} \to \mathrm{Def}(\mathrm{C})$ *to* Δ. *Then on* Δ *the following properties hold true*

(1) the sheaf of sets of the irreducible components is constant;
(2) the sheaf of sets of connected subcurves is constant.

Proof. Let us first prove (1). Consider the normalization $\widetilde{\Delta} \to \Delta$ and denote by $\mathcal{C}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ the pull-back of the family $\mathcal{C}_\Delta \to \Delta$. According to Fact 2.2, the normalization $\widetilde{\mathcal{C}}_{\widetilde{\Delta}} \to \mathcal{C}_{\widetilde{\Delta}}$ is a simultaneous normalization of the family $\mathcal{C}_{\widetilde{\Delta}} \to \widetilde{\Delta}$. In particular, the sheaf of connected components of the family $\widetilde{\mathcal{C}}_{\widetilde{\Delta}} \to \widetilde{\Delta}$, which coincides with the pull-back to $\widetilde{\Delta}$ of the sheaf of irreducible components of the family $\mathcal{C}_{\widetilde{\Delta}} \to \widetilde{\Delta}$, is locally constant on $\widetilde{\Delta}$, hence constant on each connected components of $\widetilde{\Delta}$. However, since Δ has a unique closed point, namely the central point $[\mathfrak{m}_\mathrm{C}] \in \mathrm{Spec}\, \mathrm{R}_\mathrm{C}$, each connected component of $\widetilde{\Delta}$ contains some closed point mapping to $[\mathfrak{m}_\mathrm{C}]$; hence the sheaf of connected components of $\widetilde{\mathcal{C}}_{\widetilde{\Delta}} \to \widetilde{\Delta}$ is constant. From this, we deduce that the sheaf of irreducible components of $\mathcal{C}_\Delta \to \Delta$ is also constant.

Let us now prove part (2). From (1), we have that if $\mathcal{C}_\Delta = \bigcup_{i=1}^N \mathcal{C}_\Delta^{(i)}$ is the decomposition into irreducible components, then the decomposition into irreducible components of the geometric fiber $\mathcal{C}_{\bar{t}}$ over any point $t \in \Delta(\mathrm{C})$ equals $\bigcup_{i=1}^N \mathcal{C}_{\bar{t}}^{(i)}$. For each t, we have, by Hironaka's formula [GLS, Lemma 3.3.2]

$$\delta(\mathcal{C}_{\bar{t}}) = \sum_{i=1}^N \delta(\mathcal{C}_{\bar{t}}^{(i)}) + \sum_{1 \leqslant k < l \leqslant N} |\mathcal{C}_{\bar{t}}^{(k)} \cap \mathcal{C}_{\bar{t}}^{(l)}|. \tag{9}$$

The delta invariant and the intersection numbers of the subcurves are upper semicontinuous functions in flat families. As the sum (9) is constant, we have that $\delta(\mathcal{C}_{\bar{t}}^{(i)})$ and $|\mathcal{C}_{\bar{t}}^{(k)} \cap \mathcal{C}_{\bar{t}}^{(l)}|$ don't depend on t. Assume $\bigcup_{i=1}^s \mathrm{C}^{(i)}$ is a connected subcurve of the central fibre such that, for some t, $\bigcup_{i=1}^s \mathcal{C}_{\bar{t}}^{(i)}$ is disconnected, namely, up to a renumbering, we have

$$\mathcal{C}'_{\bar{t}} \bigcap \mathcal{C}''_{\bar{t}} = \varnothing, \text{ with } \mathcal{C}'_{\bar{t}} := \left(\bigcup_{i=1}^a \mathcal{C}_{\bar{t}}^{(i)} \right), \text{ and } \mathcal{C}''_{\bar{t}} := \left(\bigcup_{i=a+1}^s \mathcal{C}_{\bar{t}}^{(i)} \right).$$

Denoting $\mathrm{C}' = \bigcup_{i=1}^{a} \mathrm{C}^{(i)}$ and $\mathrm{C}'' = \bigcup_{i=a+1}^{s} \mathrm{C}^{(i)}$, by the argument above we have $|\mathrm{C}' \cap \mathrm{C}''| = |\mathcal{C}'_{\overline{t}} \cap \mathcal{C}''_{\overline{t}}| = 0$. Since C' and C'' have no common component, their intersection number is strictly positive unless the curves are disjoint, which would contradict the connectedness of $\bigcup_{i=1}^{s} \mathrm{C}^{(i)}$. □

2.3. **Fine compactified Jacobians.** We collect results on fine compactified Jacobians of connected (reduced projective) curves with locally planar singularities and their families.

2.3.1. *Fine compactified Jacobians.* Throughout this subsubsection, we fix a connected (geometrically reduced and projective) curve C over a field k and we set $\overline{\mathrm{C}} := C \otimes_k \overline{k}$. Moreover, given a sheaf $\mathcal{I}$ on C, we denote by $\overline{\mathcal{I}}$ its pull-back to $\overline{\mathrm{C}}$.

Fine compactified Jacobians of C will parametrize certain sheaves on C, which we now introduce.

Definition 2.4. A coherent sheaf $\mathcal{I}$ on a curve C is said to be:

(i) *rank*-1 if $\overline{\mathcal{I}}$ has generic rank 1 at every irreducible component of $\overline{\mathrm{C}}$;
(ii) *torsion-free* (or pure of dimension one) if $\mathrm{Supp}(\overline{\mathcal{I}}) = \overline{\mathrm{C}}$ and every non-zero subsheaf $\mathcal{J} \subseteq \mathcal{I}$ is such that $\dim \mathrm{Supp}(\mathcal{J}) = 1$.

Note that any line bundle on C is a rank-1, torsion-free sheaf.

The construction of fine compactified Jacobians of a reducible curve C will depend on the choice of a general polarization on C, which we now introduce. We follow the notation of [MRV1].

Definition 2.5.

(i) A *polarization* on a curve C is a collection of rational numbers $\underline{m} = \{\underline{m}_{\mathrm{C}_i}\}$, one for each irreducible component C_i of $\overline{\mathrm{C}}$, such that $|\underline{m}| := \sum_i \underline{m}_{\mathrm{C}_i} \in \mathbb{Z}$. We call $|\underline{m}|$ the total degree of $\underline{m}$. Given any subcurve $D \subseteq \overline{\mathrm{C}}$, we set $\underline{m}_D := \sum_{\mathrm{C}_i \subseteq D} \underline{m}_{\mathrm{C}_i}$.
(ii) A polarization $\underline{m}$ is called *integral* at a subcurve $D \subseteq \overline{\mathrm{C}}$ if $\underline{m}_E \in \mathbb{Z}$ for any connected component E of D and of D^c. A polarization is called *general* if it is not integral at any non-trivial subcurve $D \subset \overline{\mathrm{C}}$.

Given a polarization $\underline{m}$ on C, we can define a (semi)stability condition for torsion-free, rank-1 sheaves on C. To this aim, for each subcurve D of $\overline{\mathrm{C}}$ and each torsion-free, rank-1 sheaf $\mathcal{I}$ on C, we denote by $\overline{\mathcal{I}}_D$ the quotient of the restriction $\overline{\mathcal{I}}_{|D}$ of $\overline{\mathcal{I}}$ to D modulo its biggest torsion subsheaf. It is easily seen that $\overline{\mathcal{I}}_D$ is torsion-free, rank-1 sheaf on D.

Definition 2.6. Let $\underline{m}$ be a polarization on C. Let $\mathcal{I}$ be a torsion-free rank-1 sheaf on C of degree $d = |\underline{m}|$.

(i) We say that $\mathcal{I}$ is *semistable* with respect to $\underline{m}$ (or $\underline{m}$-semistable) if for every non-trivial subcurve $D \subset \overline{\mathrm{C}}$, we have that

$$\chi(\overline{\mathcal{I}}_D) \geqslant \underline{m}_D, \tag{10}$$

where χ denotes the Euler-Poincaré characteristic.
(ii) We say that $\mathcal{I}$ is *stable* with respect to $\underline{m}$ (or $\underline{m}$-stable) if it is semistable with respect to $\underline{m}$ and if the inequality (10) is always strict.

General polarizations on C can be also characterized more geometrically:

Lemma 2.7. [MRV1, Lemmas 2.14, 5.13] *Let $\underline{m}$ be a polarization on a curve C. If $\underline{m}$ is general then every rank-1 torsion-free sheaf which is $\underline{m}$-semistable is also $\underline{m}$-stable. The converse implication is true if $\overline{\mathrm{C}}$ has locally planar singularities.*

Fine compactified Jacobians were constructed in full generality by Esteves in [Est].

Theorem 2.8 (Esteves)**.** *Let* C *be a geometrically connected curve and* $\underline{m}$ *be a general polarization on* C*. There exists a projective scheme* $\overline{J}_{\mathrm{C}}(\underline{m})$*, called the* fine compactified Jacobian of C with respect to the polarization $\underline{m}$*, which is a fine moduli space for torsion-free, rank-*1*,* $\underline{m}$*-semistable sheaves on* C*.*

Since $\underline{m}$ is general, sheaves in $\overline{J}_{\mathrm{C}}(\underline{m})$ are $\underline{m}$-stable, hence geometrically simple, by Lemma 2.7. This is the reason why $\overline{J}_{\mathrm{C}}(\underline{m})$ is a fine moduli scheme. Observe also that, clearly, we have that $\overline{J}_{\mathrm{C}}(\underline{m}) \otimes_k \overline{k} \cong \overline{J}_{\overline{\mathrm{C}}}(\underline{m})$.

We denote by $J_{\mathrm{C}}(\underline{m})$ the open subset of $\overline{J}_{\mathrm{C}}(\underline{m})$ parametrizing line bundles on C. Note that $J_{\mathrm{C}}(\underline{m})$ is isomorphic to the disjoint union of a certain number of copies of the generalized Jacobian $J_{\mathrm{C}} = \mathrm{Pic}^{\underline{0}}(\mathrm{C})$ of C.

If C has locally planar singularities and $k = \overline{k}$, its fine compactified Jacobians enjoy the following properties (see [MRV1, Thm. A]).

Theorem 2.9. *Let* C *be a connected curve with locally planar singularities over* $k = \overline{k}$ *and* $\underline{m}$ *a general polarization on* C*. Then*

- *(i)* $\overline{J}_{\mathrm{C}}(\underline{m})$ *is a connected reduced projective scheme with locally complete intersection singularities and trivial dualizing sheaf.*
- *(ii)* $J_{\mathrm{C}}(\underline{m})$ *is the smooth locus of* $\overline{J}_{\mathrm{C}}(\underline{m})$*. In particular,* $J_{\mathrm{C}}(\underline{m})$ *is dense in* $\overline{J}_{\mathrm{C}}(\underline{m})$ *and* $\overline{J}_{\mathrm{C}}(\underline{m})$ *has pure dimension equal to the arithmetic genus* $g(\mathrm{C})$ *of* C*.*
- *(iii) The number of irreducible components of* $\overline{J}_{\mathrm{C}}(\underline{m})$ *depends only on the curve* C *and not on the polarization* $\underline{m}$*.*

Therefore, the number of irreducible component of any fine compactified Jacobian of a connected curve C with locally planar singularities over $k = \overline{k}$ is an invariant of C, which is usually called the complexity of C and denoted by $c(\mathrm{C})$. We refer the reader to [MRV1, Sec. 5.1] for an explicit formula for $c(\mathrm{C})$ in terms of the intersection numbers between the subcurves of C. We just mention that if C is nodal, then $c(\mathrm{C})$ is given by the complexity of its dual graph, i.e. the number of its spanning trees.

The above Theorem (2.9) implies that any two fine compactified Jacobians of a curve C with locally planar singularities over $k = \overline{k}$ are birational Calabi-Yau (singular) varieties. However, in [MRV1, Sec. 3], the authors constructed some nodal reducible curves which do have non isomorphic (and even non homeomorphic if $k = \mathbb{C}$) fine compactified Jacobians. Despite this, the first part of our Main Theorem 1.16 implies that any two fine compactified Jacobians of a curve C with locally planar singularities have the same Betti numbers if $k = \mathbb{C}$, recovering in particular Theorem 2.9(iii). It is shown in [MRV2] and [MRV3] that all fine compactified Jacobians are derived equivalent.

2.3.2. *Relative fine compactified Jacobians.* Given a family $\pi : \mathcal{C} \to B$ of geometrically connected (and geometrically reduced) curves, i.e. a projective and flat morphism π whose geometric fiber $\mathcal{C}_{\overline{b}} := \pi^{-1}(b) \otimes_{k(b)} \overline{k(b)}$ over any point $b \in B$ is a connected (and reduced) curve, a *relative fine compactified Jacobian* for π is a scheme $\pi^J : \overline{J}_{\mathcal{C}} \to B$ projective over B, such that the geometric fiber $(\overline{J}_{\mathcal{C}})_{\overline{b}} := (\pi^J)^{-1}(b) \otimes_{k(b)} \overline{k(b)}$ over any point $b \in B$ is a fine compactified Jacobian for the curve $\mathcal{C}_{\overline{b}}$.

In the sequel, we will need the existence of relative fine compactified Jacobians for the semiuniversal deformation $\mathcal{C} \to \mathrm{Def}(\mathrm{C})$ of a geometrically connected (geometrically reduced and projective) curve C. More precisely, we have the following

Theorem 2.10. *Let C be a geometrically connected (reduced and projective) curve over k and let $\underline{m}$ be a general polarization on C. There is a relative fine compactified Jacobian $\pi^J : \overline{J}_{\mathcal{C}}(\underline{m}) \to \mathrm{Def}(\mathrm{C})$ for the semiuniversal deformation family $\pi : \mathcal{C} \to \mathrm{Def}(\mathrm{C})$ of C such that the central fiber $\overline{J}_{\mathcal{C}}(\underline{m})_0$ is isomorphic to $\overline{J}_{\mathrm{C}}(\underline{m})$.*

Furthermore, if $\overline{\mathrm{C}}$ has locally planar singularities then

(i) π^J is flat with geometric fibers of pure dimension $g(\mathrm{C})$;
(ii) $\overline{J}_{\mathcal{C}}(\underline{m})$ is regular and irreducible.

Proof. The proof in the case $k = \overline{k}$ is given in [MRV1, Thm. 5.2, Thm. 5.3], building upon the work of Esteves [Est]. However, an inspection of loc. cit. reveals that the same proof goes through in the case of an arbitrary field k. $\square$

In [MRV1, Thm. 5.2], the reader can find an explicit description of all the geometric fibers of $\pi^J : \overline{J}_{\mathcal{C}}(\underline{m}) \to \mathrm{Def}(\mathrm{C})$, which in particular shows that the above relative fine compactified Jacobian is unique.

A slight adaptation of the proof of the above Theorem 2.10 (which however we omit since it is not needed in the present work) gives the following existence results, the first one is global and the second one is local.

Proposition 2.11. *Let $\pi : \mathcal{C} \to B$ be a family of geometrically connected curves.*

1. *Up to passing to an étale cover of B, there exists a relative fine compactified Jacobian $\pi^J : \overline{J}_{\mathcal{C}} \to B$ for π.*
2. *Fix a point $b \in B$ and a general polarization $\underline{m}$ on the fiber $\mathcal{C}_b$ over b. Then, up to replacing B with an étale neighborhood of b, there exists a family of fine compactified Jacobians $\pi^J : \overline{J}_{\mathcal{C}}(\underline{m}) \to B$ such that $\overline{J}_{\mathcal{C}}(\underline{m})_b = \overline{J}_{\mathcal{C}_b}(\underline{m})$. Moreover we have (up to replacing B with an open neighborhood of b):*
   - *(i) if $\mathcal{C}_{\overline{b}}$ has locally planar singularities and B is geometrically unibranch (e.g. normal) and reduced at b, then π^J is flat with geometric fibers of pure dimension $g(\mathcal{C}_{\overline{b}})$;*
   - *(ii) if $\mathcal{C}_{\overline{b}}$ has locally planar singularities and π is versal at b, then $\overline{J}_{\mathcal{C}}(\underline{m})$ is regular.*

2.4. **Cohomology.** For the proof of our main result, Theorem 1.16 we need to reduce to positive characteristic and use some results on $l-$adic sheaves and their weights, which we now shortly review. See [BBD, Del], and [KW] for a textbook exposition. In Appendix A we give some details on the "CKS complex", which plays a major role in this paper, in the étale setting.

In this section, we work over a finite field $\mathbb{F}_\pi$, with $\pi = p^r$ and its algebraic closure $\overline{\mathbb{F}}_\pi$. For a scheme $X_{/\mathbb{F}_\pi}$, or a sheaf $\mathcal{F}$ over it, we write $\overline{X} := X \times_{\mathbb{F}_\pi} \overline{\mathbb{F}}_\pi$, $\overline{\mathcal{F}}$, etc. for their base changes to the algebraic closure.

By a "sheaf on X" we always mean a constructible $\overline{\mathbb{Q}}_\ell$-sheaf. We use the term "local system" for what is usually called a "lisse sheaf", for homogeneity with the complex case. We denote by $D^b_c(X)$ the constructible derived category, see the discussion in [BBD, §2.2], and by $\mathrm{K}(D^b_c(X))$ its Grothendieck group.

We denote by $|X|$, resp. $|\overline{X}|$ the set of closed points of X, resp. $\overline{X}$. We identify

$$|\overline{X}| = X(\overline{\mathbb{F}}_\pi) = \mathrm{Hom}(\mathrm{Spec}(\overline{\mathbb{F}}_\pi), X).$$

Given X there is a 'relative' Frobenius morphism $Fr : \overline{X} \to \overline{X}$ of schemes over $\overline{\mathbb{F}}_\pi$, and, for every r, the set $X(\mathbb{F}_{\pi^r})$ of $\mathbb{F}_{\pi^r}$-valued points is identified with the fixed points of Fr^r in $\overline{X}$.

Given $K \in D^b_c(X)$ on X, its pull-back $\overline{K}$ to $\overline{X}$ is endowed with an isomorphism $F^* : Fr^*\overline{K} \to \overline{K}$, inducing a morphism $F^* : H^*(\overline{X}, \overline{K}) \to H^*(\overline{X}, \overline{K})$, and, since Frobenius is a proper map, $F^* : H^*_c(\overline{X}, \overline{K}) \to H^*_c(\overline{X}, \overline{K})$.

For $x \in X(\mathbb{F}_{\pi^r}) = (\overline{X})^{Fr^r}$, and $\mathcal{F}$ a sheaf on X, there is an induced isomorphism

$$(F^r_x)^* : \overline{\mathcal{F}}_{\overline{x}} \to \overline{\mathcal{F}}_{\overline{x}},$$

where $\overline{x}$ is a geometric point lying over x. Its trace is independent of the choice of the geometric point and will be denoted $tr((F^r_x)^* : \overline{\mathcal{F}}_x \to \overline{\mathcal{F}}_x)$. In general we will write simply "tr" to indicate, when applied to a *graded* endomorphism, the usual alternating sum of traces, namely, for an endomorphism $A = \oplus A_i$ of a graded vector space $V = \oplus V_i$, we set

$$\mathrm{tr} A = \sum_i (-1)^i \mathrm{tr}(A_i : V_i \to V_i).$$

.

The Grothendieck-Lefschetz trace formula asserts that, for

$$\mathrm{tr}(F^* : H^*_c(\overline{X}, \overline{K}) \to H^*_c(\overline{X}, \overline{K})) = \sum_{x \in X(\mathbb{F}_\pi)} \mathrm{tr}(F^*_x : \mathcal{H}(\overline{K})_x \to \mathcal{H}(\overline{K})_x), \tag{11}$$

where by $\mathcal{H}(\overline{K})_x$ we mean the graded vector space $\oplus_i \mathcal{H}^i(\overline{K})_x[-i]$. In particular when $\mathcal{F} = \overline{\mathbb{Q}}_\ell$, the formula asserts that the number of points over $\mathbb{F}_\pi$ is the trace of Frobenius on the compactly supported cohomology of X. Clearly a similar formula holds for the powers of the Frobenius map. The 'sheaf-function correspondence' collects the pointwise traces of a sheaf:

$$\begin{array}{rcl} [K] : \coprod_n X(\mathbb{F}_{\pi^n}) & \to & \overline{\mathbb{Q}}_\ell \\ x \in X(\mathbb{F}_{\pi^n}) & \mapsto & \mathrm{tr}((F^n_x)^* : \overline{K}_x \to \overline{K}_x). \end{array}$$

The map $K \to [K]$ factors through $\mathrm{K}(D^b_c(X))$. According to [Lau, Thm. 1.1.2], in fact it determines an injection from $\mathrm{K}(D^b_c(X))$ to collections of functions on the $X(\mathbb{F}_{\pi^n})$.

We recall the formalism of weights. Fix an embedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$; we use $|\cdot|$ to denote the absolute value of $\mathbb{C}$. The 'weight' of an element $\xi \in \overline{\mathbb{Q}}_\ell$ is by definition $2\log_\pi |\xi|$. An endomorphism of an $\overline{\mathbb{Q}}_\ell$ vector space is said to be pure of weight w if all its eigenvalues are. A sheaf $\mathcal{F}$ on X is *pointwise pure* of weight w if, for any closed point $x \in |X|$, the endomorphism $(F^{\deg(x)})^* : \overline{\mathcal{F}}_x \to \overline{\mathcal{F}}_x$ is pure of weight $\deg(x)w$. (E.g., the constant sheaf $\overline{\mathbb{Q}}_\ell$ is pointwise pure of weight zero, and the Tate sheaf $\overline{\mathbb{Q}}_\ell(1)$ is pointwise pure of weight -2.)

A sheaf $\mathcal{F}$ is said to be *mixed of weight* $\leqslant w$ if it is an iterated extension of sheaves which are pointwise pure of weight $\leqslant w$ and $K \in D^b_c(X)$ is said to be mixed of weight $\leqslant w$ if $\mathcal{H}^i(K)$ is mixed of weight $\leqslant i + w$. (E.g., $\overline{\mathbb{Q}}_\ell$ is mixed of weight $\leqslant 0$, and $\overline{\mathbb{Q}}_\ell[1]$ is mixed of weight $\leqslant 1$.)

Remark 2.12. Since all sheaves and complexes of sheaves which we encounter are mixed, we will still denote by $D^b_c(X)$ the triangulated subcategory of the constructible category consisting of mixed complexes. The stability properties of this category are stated in [BBD, 5.1.6 and 5.1.7]. For the sake of notational simplicity we will denote by $f_*, f_!$, instead of $Rf_*, Rf_!$ the direct image functors.

An object in $D_c^b(X)$ is *pure of weight* w if it is mixed of weight $\leqslant w$, and its Verdier dual is mixed of weight $\leqslant -w$. This notion is different from that of being pointwise pure. For example, the constant sheaf $\mathbb{Q}_\ell$ is always pointwise pure of weight 0. On a smooth space, it is also pure: it is certainly mixed of weight $\leqslant 0$, and $D\mathbb{Q}_\ell = \mathbb{Q}_\ell[2\dim X](\dim X)$ is pointwise pure of weight $2\dim X - 2\dim X = 0$. However, for a general space X, purity of $\mathbb{Q}_\ell$ is equivalent to the assertion that $\mathbb{Q}_\ell$ is the intersection cohomology sheaf of X.

The following facts about weights are hard to prove ([Del]) but easy to use. Given any map $f : X \to Y$, if $K \in D_c^b(X)$ is mixed of weight $\leqslant w$, then so is $f_! K$. It follows by a duality argument that if K is mixed of weight $\geqslant w$ then so is $f_* K$ ([BBD, 5.1.14]).

By [BBD, Cor. 5.3.4], every simple perverse sheaf is pure, and every pure complex of sheaves becomes a sum of shifted simple perverse sheaves after pulling back to $\overline{X}$. We will require the following variant of the statement that the trace function of a sheaf determines its class in $\mathrm{K}(D_c^b(X))$:

Lemma 2.13. *Let $K, L \in D_c^b(X)$ be pure of the same weight w, and sums of shifts of simple perverse sheaves. If $[K] = [L]$, then $\overline{K} \cong \overline{L}$.*

Proof. By Noetherian induction, it suffices to treat the case where X is smooth and K, L are sums of shifted semi-simple local systems, say $K = \bigoplus K_a[-a]$ and $L = \bigoplus L_a[-a]$. For $x \in X(\mathbb{F}_{\pi^m})$, the function $[K]$ determines the set of eigenvalues of Fr_x^m. The ones belonging to K_a are those of absolute value $\pi^{(w+a)/2}$, hence we may recover $[K_a]$ for every a. By hypothesis we have $[K_a] = [L_a]$, which, by a standard argument relying on Chebotarev theorem (see [Lau, Prop. 1.1.2]) implies $K_a \cong L_a$ for every a. ☐

3. Nodal Curves

3.1. **Notation.** In this section our curve C is a (reduced) nodal curve, defined over a finite field $k = \mathbb{F}_\pi$, and $\overline{\mathrm{C}} := \mathrm{C} \otimes_k \overline{k}$.

The statements in this section have an obvious counterpart if the curve is defined over the field of complex numbers and we consider the weight formalism coming from Mixed Hodge theory. Let ℓ be a prime number other than the characteristic of k; we work throughout this section with étale cohomology with coefficients in $\overline{\mathbb{Q}}_\ell$.

3.1.1. *The dual graph.* We write $\Gamma = \Gamma_{\overline{\mathrm{C}}}$ for the dual graph of the curve $\overline{\mathrm{C}}$: its vertices $v \in \mathrm{V}$ correspond to the irreducible components of $\overline{\mathrm{C}}$, and its edges $e \in \mathrm{E}$ correspond to the nodes of $\overline{\mathrm{C}}$. Note that, since we do not assume C geometrically connected, Γ may be disconnected. The Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on the graph Γ, and in particular on the sets E and V.

We write $\mathbb{V} = \mathbb{V}_\Gamma := C_0(\Gamma, \overline{\mathbb{Q}}_\ell)$ and $\mathbb{E} = \mathbb{E}_\Gamma := C_1(\Gamma, \overline{\mathbb{Q}}_\ell)$ for the $\mathrm{Gal}(\overline{k}/k)$-modules of zero- and one-simplicial chains on Γ. Explicitly, $\mathbb{V}$ is the $\overline{\mathbb{Q}}_\ell$-vector space of $\overline{\mathbb{Q}}_\ell$-linear combination of vertices of Γ and $\mathbb{E}$ is the $\overline{\mathbb{Q}}_\ell$-vector space of $\overline{\mathbb{Q}}_\ell$-linear combination of oriented edges of Γ modulo the relation $\vec{e} = -\overleftarrow{e}$, where $\vec{e}$ and $\overleftarrow{e}$ denote the two oriented edges corresponding to an (unoriented) edge e of Γ. The actions of $\mathrm{Gal}(\overline{k}/k)$ on $\mathbb{V}$ and $\mathbb{E}$ are induced by the action on V and E so that $\mathbb{V}$ is a permutation representation while $\mathbb{E}$ is only a signed permutation representation (because the Galois action can reverse the oriented edges of Γ). The homology of the graph Γ is defined via the following exact sequence

$$0 \to H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \to \mathbb{E} \xrightarrow{\partial} \mathbb{V} \to H_0(\Gamma, \overline{\mathbb{Q}}_\ell) \to 0, \tag{12}$$

where ∂ is the boundary map which sends an oriented edge into the difference between its target and its source.

We write $\mathbb{V}^* = C^0(\Gamma, \overline{\mathbb{Q}}_\ell)$ and $\mathbb{E}^* = C^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ for the dual $\mathrm{Gal}(\overline{k}/k)$-modules of zero- and one-simplicial cochains on Γ. Since $\mathbb{V}$ and $\mathbb{E}$ are both signed permutation representations, there are isomorphisms of $\mathrm{Gal}(\overline{k}/k)$-modules $\mathbb{E} \cong \mathbb{E}^*$ and $\mathbb{V} \cong \mathbb{V}^*$. The cohomology of Γ is defined by mean of the following exact sequence

$$0 \to H^0(\Gamma, \overline{\mathbb{Q}}_\ell) \to \mathbb{V}^* \xrightarrow{\partial^*} \mathbb{E}^* \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \to 0, \tag{13}$$

where ∂^* is the dual of the map ∂.

When k is a finite field, we will want the weights of the Frobenius action on $\mathbb{E}$, $\mathbb{V}$, $\mathbb{E}^*$, $\mathbb{V}^*$, $H_0(\Gamma, \overline{\mathbb{Q}}_\ell)$, $H_1(\Gamma, \overline{\mathbb{Q}}_\ell)$, $H^0(\Gamma, \overline{\mathbb{Q}}_\ell)$, $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$. Since in each case, the $\mathrm{Gal}(\overline{k}/k)$ action factors through a finite group, all these spaces are pure of weight zero.

3.1.2. *Geometric interpretation of the cohomology of the dual graph.* These vector spaces arise geometrically from curves related to $\overline{\mathrm{C}}$ by normalization and deformation.

Cohomology of the graph comes from the normalization $\nu : \overline{\mathrm{C}}^\nu \to \overline{\mathrm{C}}$. The sequence of sheaves

$$0 \to \overline{\mathbb{Q}}_\ell \to \nu_* \overline{\mathbb{Q}}_\ell \to \nu_* \overline{\mathbb{Q}}_\ell / \overline{\mathbb{Q}}_\ell \to 0$$

yields by taking cohomology:

$$0 \to H^0(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}, \nu_* \overline{\mathbb{Q}}_\ell / \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to 0.$$

We have defined $\mathbb{V}^*, \mathbb{E}^*$ so as to have canonical, $\mathrm{Gal}(\overline{k}/k)$-equivariant identifications

$$\begin{aligned} \mathbb{V}^* &= H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell), \\ \mathbb{E}^* &= H^0(\overline{\mathrm{C}}, \nu_* \overline{\mathbb{Q}}_\ell / \overline{\mathbb{Q}}_\ell) \end{aligned}$$

Substituting in $H^1(\Gamma, \overline{\mathbb{Q}}_\ell) = \mathrm{Cok}(H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}, \nu_* \overline{\mathbb{Q}}_\ell / \overline{\mathbb{Q}}_\ell))$, we find the short exact sequence

$$0 \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to 0,$$

which, since $H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ is pure of weight zero and $H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)$ is pure of weight one, gives the weight filtration of $H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell)$.

On the other hand, homology of the graph comes from a one-parameter smoothing $\sigma : \mathcal{C} \to \mathbb{D}$ of C, with special fibre $\mathcal{C}_0 = \overline{\mathrm{C}}$ and geometric generic fibre $\mathcal{C}_{\overline{\eta}}$. The cohomology of the nearby-vanishing sequence gives:

$$0 \to H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}, \Phi_\sigma \overline{\mathbb{Q}}_\ell) \to H^2(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^2(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to 0. \tag{14}$$

By Poincaré duality we have

$$H^2(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) = H^2(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \cong H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)^* \otimes \mathbb{L} = \mathbb{V} \otimes \mathbb{L}$$

and, likewise

$$H^2(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \cong H^0(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)^* \otimes \mathbb{L} \cong H^0(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell)^* \otimes \mathbb{L}.$$

Finally, we have by Picard-Lefschetz formula ([Mil], p.207):

$$H^1(\overline{\mathrm{C}}, \Phi_\sigma \overline{\mathbb{Q}}_\ell) \cong \mathbb{E} \otimes \mathbb{L}.$$

Substituting in formula (14) we find:

$$0 \to H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} \to 0.$$

The (monodromy-)weight filtration on $H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ is:

$$\begin{aligned} W_0 H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H^1(\Gamma, \overline{\mathbb{Q}}_\ell), \\ W_1 H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell), \\ W_2 H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell), \end{aligned}$$

with associated graded pieces

$$\begin{aligned} Gr_0^W H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H^1(\Gamma, \overline{\mathbb{Q}}_\ell), \\ Gr_1^W H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell), \\ Gr_2^W H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) &= H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}. \end{aligned}$$

3.1.3. *Subgraphs and partial normalizations.* For every subset $I \subset \mathrm{E}$, we factor the normalization map

$$\overline{\mathrm{C}}^\nu \xrightarrow{\nu^I} \overline{\mathrm{C}}^I \xrightarrow{\nu_I} \overline{\mathrm{C}}$$

where $\nu_I : \overline{\mathrm{C}}^I \to \overline{\mathrm{C}}$ is the partial normalization of the nodes of the subset I, and $\nu^I : \overline{\mathrm{C}}^\nu \to \overline{\mathrm{C}}^I$ for the remaining normalization. We have sequences

$$0 \to H^0(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}^I, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}, \nu_{I*}\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}^I, \overline{\mathbb{Q}}_\ell) \to 0$$

and

$$0 \to H^0(\overline{\mathrm{C}}^I, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to H^0(\overline{\mathrm{C}}^I, \nu^I_*\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}^I, \overline{\mathbb{Q}}_\ell) \to H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \to 0.$$

The dual graph of the partial normalization $\overline{\mathrm{C}}^I$ is the graph $\Gamma \backslash I$, which is obtained from $\Gamma = \Gamma_{\mathrm{C}}$ by deleting the edges corresponding to I. As in §3.1.2, we have canonical identifications $\mathbb{E}^*_{\Gamma\backslash I} = H^0(\overline{\mathrm{C}}^I, \nu^I_*\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}}_\ell)$ and $\mathbb{V}^*_{\Gamma\backslash I} = H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) = \mathbb{V}^*_\Gamma = \mathbb{V}^*$. Moreover, we set $\mathbb{E}^*_I = H^0(\overline{\mathrm{C}}, \nu_{I*}\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}}_\ell)$ so that we have a canonical splitting $\mathbb{E}^* = \mathbb{E}^*_\Gamma = \mathbb{E}^*_{\Gamma\backslash I} \oplus \mathbb{E}^*_I$.

We now introduce a collection of subsets of E which will play an important role in what follows.

Definition 3.1. We write $\mathscr{C}(\Gamma)$ for the collection of subsets of E whose removal disconnects no component of Γ, i.e. a subset $I \subseteq \mathrm{E}$ belongs to $\mathscr{C}(\Gamma)$ if and only if $\Gamma\backslash I$ has the same number of connected components of Γ.

We set $n_i(\Gamma) := \#\{I \in \mathscr{C}(\Gamma) \mid \dim H_1(\Gamma\backslash I) = i\}$.

Note that $n_0(\Gamma)$, i.e. the cardinality of the set of maximal elements of $\mathscr{C}(\Gamma)$, is also equal to the complexity $c(\Gamma)$ of Γ, i.e. the number of spanning forests of Γ.

An alternative characterization of the elements of $\mathscr{C}(\Gamma)$ is provided by the following

Lemma 3.2. *A subset $I \subseteq \mathrm{E}$ belongs to $\mathscr{C}(\Gamma)$ iff the composition $\mathbb{E}^*_I \to \mathbb{E}^* \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ is injective. In that case, the following sequence is exact:*

$$0 \to \mathbb{E}^*_I \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \to H^1(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell) \to 0.$$

Proof. The inclusion of graphs $\Gamma\backslash I \hookrightarrow \Gamma$ induces a pull-back map from the sequence (13) to the analogous sequence for $\Gamma\backslash I$. Applying the snake lemma to this map of sequences and using that $\mathbb{V}^*_{\Gamma\backslash I} = \mathbb{V}^*_\Gamma$, we get the exact sequence

$$0 \to H^0(\Gamma, \overline{\mathbb{Q}}_\ell) \to H^0(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell) \to \mathbb{E}^*_I \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \to H^1(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell) \to 0.$$

By Definition 3.1, the subset I belongs to $\mathscr{C}(\Gamma)$ if and only if the map $H^0(\Gamma, \overline{\mathbb{Q}}_\ell) \to H^0(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell)$ is an isomorphism. By the above exact sequence, this happens precisely when the map $\mathbb{E}^*_I \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ is injective and in that case we get the required short exact sequence. □

Remark 3.3. It follows from Lemma 3.2 that $\mathscr{C}(\Gamma)$ is the collection of all subsets of E whose images under the map $\mathbb{E}^* \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ remain linearly independent. Thus $\mathscr{C}(\Gamma)$ is the collection of independent elements of a (representable) matroid – in particular, a simplicial complex – which is usually called the cographic matroid of the graph Γ.

Fixing orientations of each edge $e \in \mathrm{E}$ of Γ and an ordering on E determines, for all $I \subset \mathrm{E}$, 'volume' elements $e_I^* \in \wedge^{|I|}\mathbb{E}_I^*$, well defined up to a sign. Lemma 3.2 may be reformulated as the assertion that $I \in \mathscr{C}(\Gamma)$ if and only if the image of e_I^* in $\wedge^{|I|}H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$ is non-zero. Indeed, even more is true as the following Lemma shows.

Lemma 3.4. *If $I \in \mathscr{C}(\Gamma)$, there is an injective map, well-defined up to a sign,*

$$\wedge e_I^* : \wedge^{i-|I|}H^1(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell) \to \wedge^i H^1(\Gamma, \overline{\mathbb{Q}}_\ell).$$

Proof. The map is defined by lifting $\eta \in \wedge^{i-|I|}H^1(\Gamma\backslash I, \overline{\mathbb{Q}}_\ell)$ arbitrarily to an element in $\wedge^{i-|I|}H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$, and then wedging by e_I^*. This is well defined because the ambiguity in the lift is killed by $\wedge e_I^*$. $\square$

3.2. **Counting points on the Hilbert schemes of points.** The aim of this section is to determine the zeta function of the Hilbert schemes of a nodal curve over a finite field.

First we recall how the Hilbert scheme factors, in the Grothendieck ring of varieties, into local contributions. Let for the moment C be any reduced curve over any field k. Let $\mathrm{C}^{[n]} \to \mathrm{C}^{(n)}$ be the Hilbert-Chow morphism from the Hilbert scheme to the symmetric product. For a point $p \in \mathrm{C}$, we write $\mathrm{C}_p^{[n]}$ for the preimage of $n \cdot [p]$ under the Hilbert-Chow morphism.

For any finite collection of points $\mathcal{P} \subset \mathrm{C}$, we have the following equality of series:

$$\sum_{n=0}^{\infty} q^n \cdot \mathrm{C}^{[n]} = \left(\sum_{n=0}^{\infty} q^n \cdot (\mathrm{C}\backslash\mathcal{P})^{[n]}\right) \cdot \prod_{p\in\mathcal{P}} \left(\sum_{n=0}^{\infty} q^n \cdot \mathrm{C}_p^{[n]}\right) \tag{15}$$

For our purposes, it will suffice to know this equality at the level of zeta equivalence, i.e. to see that both sides have the same number of points over finite fields. For this, it is enough to observe that every point in $\mathrm{C}^{[n]}$ is the union of a subscheme supported on $\mathcal{P}$ and a subscheme supported off $\mathcal{P}$. But in fact, a standard fpqc descent argument shows that we have an identity in the Grothendieck ring of varieties $K_0(Var/k)[[q]]$.

We now turn to the problem at hand. Let C be a nodal curve over $\mathbb{F}_\pi$, and let $\nu : \mathrm{C}^\nu \to \mathrm{C}$ be its normalization. Since we are interested in counting points, let $\overline{\mathrm{C}}^\nu$ and $\overline{\mathrm{C}}$ be the corresponding curves over the algebraic closure. We write E for the singular set of $\overline{\mathrm{C}}$, since these points are naturally identified with the set of edges in the dual graph. We write $\mathrm{E}^\nu := \nu^{-1}(\mathrm{E})$ for its preimage; note this is naturally identified with the oriented edges of the dual graph.

In what follows we work with the Grothendieck group of $\mathbb{F}_\pi$ varieties up to zeta-equivalence, which in practice means that all equalities are asserting $\mathrm{Gal}(\overline{\mathbb{F}}_\pi/\mathbb{F}_\pi)$-equivariant bijections of sets of $\overline{\mathbb{F}}_\pi$-points.

We first analyze the Hilbert schemes of the normalization; as this curve is smooth, these are just symmetric powers. Nonetheless we subject them to equation 15.

$$\begin{aligned}
\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}^\nu)^{[n]} &= \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}^\nu\backslash\mathrm{E}^\nu)^{[n]}\right) \cdot \prod_{\vec{e}\in\mathrm{E}^\nu} \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}_{\vec{e}}^\nu)^{[n]}\right) \\
&= \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}^\nu\backslash\mathrm{E}^\nu)^{[n]}\right) \cdot \prod_{\vec{e}\in\mathrm{E}^\nu} \left(\frac{1}{1 - q\cdot[\vec{e}]}\right)
\end{aligned}$$

Note that the Galois group acts on E^ν, and that the above equation is Galois-equivariant. Again, this means that both sides are formal power series of sets on which the Galois group acts, and the above equation asserts a bijection.

We turn to the Hilbert schemes of $\overline{\mathrm{C}}$. We have

$$\sum_{n=0}^{\infty} q^n \cdot \overline{\mathrm{C}}^{[n]} = \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}\backslash \mathrm{E})^{[n]}\right) \cdot \prod_{e\in \mathrm{E}} \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}_e)^{[n]}\right)$$

Fix some $e \in \mathrm{E}$; we will study $\mathrm{C}_e^{[n]}$. We have naturally $\mathrm{C}_e^{[0]} = [\mathrm{Spec}\, \mathbb{F}_\pi] = 1$ and $\mathrm{C}_e^{[1]} = [e]$. Forgetting for the moment the Galois action, the Hilbert schemes of points on nodal curves are explicitly described in, for instance, [Har, Prop. 3.1] and [Ran, Thm. 1]. For $n \geqslant 2$, the space $\mathrm{C}_e^{[n]}$ consists of a chain of $(n-1)$ rational curves, meeting at the points which correspond to the monomial ideals (x^a, y^{n-a+1}) for $a = 2, \ldots, n-1$. The nonsingular points of the i-th component correspond to the ideals $(y^i + \alpha x^{n-i})$ for $\alpha \in \mathbb{G}_m$, plus the two remaining monomial ideals (x^n, y) and (x, y^n).

Now remembering the Galois action, note that the n monomial ideals above are naturally identified with $Sym^{n-1}(e^\nu)$, where $e^\nu = \{\overleftarrow{e}, \overrightarrow{e}\}$ denotes the two-point set of preimages of e under the normalization map. Similarly, the complement of these points is naturally identified with $Sym^{n-2}(e^\nu) \cdot [\mathbb{G}_m]$. The meaning of such assertions is always that the equality is natural in these names and so when we plug it back into Equation 15, the symbols will have their evident meaning in terms of the Galois action.

For the purpose of studying the Galois orbits, $[e]^k$ all behave identically for $k \geqslant 1$. Similarly, since $\overleftarrow{e} \to e$ comes from a map defined over $\mathbb{F}_\pi$, we can always replace $[\overleftarrow{e}]$ with $[\overleftarrow{e}][e]$. We also have $[\overleftarrow{e}][\overrightarrow{e}] = [e]$. Finally, we introduce the symbol $\chi_e := [e^\nu] - [e]$. Note that, just as the e naturally form the set E, the χ_e are naturally the basis of $\mathbb{E}$.

We compute:

$$\begin{aligned}
\sum_{n=0}^{\infty} q^n \cdot (\mathrm{C}_e)^{[n]} &= 1 + q\cdot[e] + \sum_{n=2}^{\infty} q^n \cdot Sym^{n-1}(e^\nu) + [\mathbb{G}_m] \cdot \sum_{n=2}^{\infty} q^n \cdot Sym^{n-2}(e^\nu) \\
&= 1 + \frac{q\cdot[e]}{(1-q\cdot[\overleftarrow{e}])(1-q\cdot[\overrightarrow{e}])} + \frac{q^2\cdot \mathbb{G}_m \cdot [e]}{(1-q\cdot[\overleftarrow{e}])(1-q\cdot[\overrightarrow{e}])} \\
&= \frac{1 - q\cdot([e^\nu]-[e]) + q^2\cdot[e]\cdot\mathbb{L}}{(1-q\cdot[\overleftarrow{e}])(1-q\cdot[\overrightarrow{e}])} \\
&= \frac{1 - q\cdot\chi_e + q^2\cdot[e]\cdot\mathbb{L}}{(1-q\cdot[\overleftarrow{e}])(1-q\cdot[\overrightarrow{e}])}
\end{aligned}$$

Doing so, we are left with the equations

$$\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}^\nu)^{[n]} = \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}^\nu\backslash \mathrm{E}^\nu)^{[n]}\right) \cdot \prod_{\vec{e}\in \mathrm{E}^\nu} \left(\frac{1}{1-q\cdot[\vec{e}]}\right)$$

and

$$\sum_{n=0}^{\infty} q^n \cdot \overline{\mathrm{C}}^{[n]} = \left(\sum_{n=0}^{\infty} q^n \cdot (\overline{\mathrm{C}}\backslash \mathrm{E})^{[n]}\right) \cdot \prod_{e\in \mathrm{E}} \left(\frac{1 - q\cdot\chi_e + q^2\cdot[e]\cdot\mathbb{L}}{(1-q\cdot[\overleftarrow{e}])(1-q\cdot[\overrightarrow{e}])}\right)$$

Dividing the second by the first and observing $\overline{\mathrm{C}}^\nu\backslash \mathrm{E}^\nu = \overline{\mathrm{C}}\backslash\mathrm{E}$ yields:

Proposition 3.5. *Let* C *be a nodal curve over* k. *Then we have an equality in the Grothendieck ring* $K_0^Z(Var/k)$ *of varieties over* k *modulo zeta equivalence:*

$$\frac{\sum q^n \mathrm{C}^{[n]}}{\sum q^n (\mathrm{C}^\nu)^{[n]}} = \prod_{e \in \mathrm{E}} \left(1 - q\chi_e + q^2 \cdot [e] \cdot \mathbb{L}\right)$$

Remark 3.6. This proposition can also be deduced from the upcoming Theorem 3.13 which determines the Frobenius traces on the stalks of the IC sheaves given by extending the local systems of cohomologies from the smooth locus of the family of compactified Jacobians $\pi^J : \overline{\mathcal{J}} \to \mathrm{Def}(\mathrm{C})$. Indeed, since the calculation of Hilbert scheme series is local at the singular points, to prove Proposition 3.5 it suffices to check it for *irreducible* curves. But by [MS, MY], for these it follows from an analogous result for the Jacobian, which can be deduced from Theorem 3.13 plus a support theorem for families of Jacobians of nodal curves. The necessary support theorem is available from the general result of [Ngô], or [MS], or direct calculation as in [dCHM].

3.3. **Fine compactified Jacobians.** In this subsection, we work over an algebraically closed field $k = \overline{k}$. Our goal is to determine the class of a fine compactified Jacobian of the nodal curve $\mathrm{C} = \overline{\mathrm{C}}$ in $K_0(Var_{\overline{k}})$.

Let us first compute the class in $K_0(Var_{\overline{k}})$ of the generalized Jacobian J_C of C, which is by definition the connected component of the Picard scheme $\mathrm{Pic}(C)$ of C containing the identity. The normalization morphism $\nu : \mathrm{C}^\nu \to \mathrm{C}$ induces the sequence

$$1 \to \mathbb{G}_m \to \nu_* \mathbb{G}_m \to \nu_* \mathbb{G}_m / \mathbb{G}_m \to 1,$$

which yields by taking cohomology:

(16)
$$1 \to H^0(\mathrm{C}, \mathbb{G}_m) \to H^0(\mathrm{C}^\nu, \mathbb{G}_m) \to H^0(\mathrm{C}, \nu_* \mathbb{G}_m/\mathbb{G}_m) \to H^1(\mathrm{C}, \mathbb{G}_m) \to H^1(\mathrm{C}^\nu, \mathbb{G}_m) \to 1.$$

In terms of the dual graph $\Gamma = \Gamma_\mathrm{C}$ of C, we have

$$1 \to H^0(\Gamma, \mathbb{Z}) \otimes \mathbb{G}_m \to H^0(\mathrm{C}^\nu, \mathbb{G}_m) \to H^0(\mathrm{C}, \nu_* \mathbb{G}_m/\mathbb{G}_m) \to H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}_m \to 1.$$

Substituting into (16) and restricting to the connected component of the identity gives an exact sequence of algebraic groups

$$1 \to H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}_m \cong \mathbb{G}_m^{h^1(\Gamma)} \to J_\mathrm{C} \xrightarrow{\nu^*} J_{\mathrm{C}^\nu} \to 1, \tag{17}$$

where $h^1(\Gamma)$ is the rank of the free abelian group $H^1(\Gamma, \mathbb{Z})$.

Since $\mathbb{G}_m = \mathrm{GL}_1$ is a special group, the sequence (17) is Zariski locally trivial, hence we have the following equality in $K_0(Var_{\overline{k}})$:

$$J_\mathrm{C} = J_{\mathrm{C}^\nu} \mathbb{G}_m^{h^1(\Gamma)} = J_{\mathrm{C}^\nu} (\mathbb{L} - 1)^{h^1(\Gamma)}. \tag{18}$$

In order to compute the class in $K_0(Var_{\overline{k}})$ of a fine compactified Jacobian $\overline{J}_\mathrm{C}(\underline{m})$ of C, we need to recall the stratification of $\overline{J}_\mathrm{C}(\underline{m})$ in terms of partial normalizations of C studied in [MV] (see also [OS, Ale]). Given any torsion free, rank-1 sheaf $\mathcal{I}$ on C, its endomorphism sheaf $\underline{\mathrm{End}}_{\mathcal{O}_\mathrm{C}}(\mathcal{I})$ is a sheaf of finite $\mathcal{O}_\mathrm{C}$-algebras such that $\mathcal{O}_\mathrm{C} \subseteq \underline{\mathrm{End}}_{\mathcal{O}_\mathrm{C}}(\mathcal{I}) \subseteq \mathcal{O}_{\mathrm{C}^\nu}$. The sheaf $\mathcal{I}$ is naturally a sheaf on the partial normalization $\mathrm{C}^\mathcal{I} := \underline{\mathrm{Spec}}_\mathrm{C}(\mathrm{End}_{\mathcal{O}_\mathrm{C}}(\mathcal{I}))$ of C; the original $\mathcal{I}$ being recovered by the pushforward along the partial normalization morphism $\nu_\mathcal{I} : \mathrm{C}^\mathcal{I} \to \mathrm{C}$. Since C is nodal, it can be checked that $\mathrm{C}^\mathcal{I}$ is the partial normalization of C at all the nodes where $\mathcal{I}$ is not locally free and $\mathcal{I}$

is a line bundle on $\mathrm{C}^{\mathcal{I}}$. This gives rise to a stratification of any fine compactified Jacobian $\overline{J}_{\mathrm{C}}(\underline{m})$ into locally closed subsets

$$\overline{J}_{\mathrm{C}}(\underline{m}) = \coprod_{S\subset \mathrm{C}_{\mathrm{sing}}} \overline{J}_{\mathrm{C},S}(\underline{m}) := \coprod_{S\subseteq \mathrm{C}_{\mathrm{sing}}} \{\mathcal{I}\in \overline{J}_{\mathrm{C}}(\underline{m}) \ : \ C^{\mathcal{I}} = C^S\}. \tag{19}$$

The following result describes the stratum $\overline{J}_{\mathrm{C},S}(\underline{m})$ in terms of the graph $\Gamma\backslash S$ obtained from the dual graph $\Gamma = \Gamma_{\mathrm{C}}$ of C by deleting the edges corresponding to S.

Proposition 3.7. *([MV, Thm. 5.1]) Let C be a connected nodal curve over $\overline{k}$ and let $\overline{J}_{\mathrm{C}}(\underline{m})$ be a fine compactified Jacobian. Then for every $S\subseteq \mathrm{C}_{\mathrm{sing}}$, the stratum $\overline{J}_{\mathrm{C},S}(\underline{m})$ is isomorphic to a disjoint union of $\hat{c}(\Gamma\backslash S)$ copies of J_{C^S}, where*

$$\hat{c}(\Gamma\backslash S) = \begin{cases} c(\Gamma\backslash S) = \#\{\textit{spanning trees of } \Gamma\backslash S\} & \textit{if } \Gamma\backslash S \textit{ is connected,}\\ 0 & \textit{if } \Gamma\backslash S \textit{ is not connected.}\end{cases} \tag{20}$$

We are now ready to compute the class of a fine compactified Jacobian of a nodal curve in $K_0(Var_{\overline{k}})$.

Proposition 3.8. *Let C be a connected nodal curve over $\overline{k}$ and let $\overline{J}_{\mathrm{C}}$ be a fine compactified Jacobian of C. Then, in $K_0(Var_{\overline{k}})$, we have*

$$\overline{J}_{\mathrm{C}}(\underline{m}) = J_{\mathrm{C}^\nu}\cdot c(\Gamma)\mathbb{L}^{h^1(\Gamma)}.$$

Proof. From the stratification (19) together with Proposition 3.7 and (18), we get that

$$\overline{J}_{\mathrm{C}}(\underline{m}) = \sum_{S\subset \mathrm{E}} \hat{c}(\Gamma\backslash S)\cdot J_{\mathrm{C}^S} = J_{\mathrm{C}^\nu}\sum_{S\subset \mathrm{E}} \hat{c}(\Gamma\backslash S)\cdot (\mathbb{L}-1)^{h^1(\Gamma\backslash S)}.$$

Thus our goal is to prove

$$\hat{c}(\Gamma)\mathbb{L}^{h^1(\Gamma)} = \sum_{S\subset \mathrm{E}} \hat{c}(\Gamma\backslash S)\cdot (\mathbb{L}-1)^{h^1(\Gamma\backslash S)}.$$

Note that if $\hat{c}(\Gamma\backslash S)$ is not zero, i.e. if $\Gamma\backslash S$ is connected, then $h^1(\Gamma\backslash S) = h^1(\Gamma) - |S|$. We substitute $x+1=\mathbb{L}$. Then the above required formula reads

$$\hat{c}(\Gamma)\sum_{i=0}^{h^1(\Gamma)}\binom{h^1(\Gamma)}{i}x^i = \sum_{S\subset \mathrm{E}} \hat{c}(\Gamma\backslash S)\cdot x^{h^1(\Gamma)-|S|}.$$

This holds for each coefficient of x by the following Lemma 3.9. □

Lemma 3.9. *For any connected graph Γ,*

$$\sum_{\substack{S\subseteq \mathrm{E}(\Gamma)\\ |S|=i}} \hat{c}(\Gamma\backslash S) = \binom{b_1(\Gamma)}{i}\cdot \hat{c}(\Gamma).$$

Proof. The LHS counts the number of ways to first remove i edges from Γ, and then find a spanning tree of Γ from what remains, whereas the RHS counts the number of ways to first find a spanning tree of Γ, which amounts to removing some $b_1(\Gamma)$ edges, and then decide which i of those edges you removed 'first'. □

From the above Proposition, we can compute the weight polynomial of fine compactified Jacobians of nodal curves.

Corollary 3.10. *Same assumptions as in Proposition 3.8. Then the weight polynomial of $\overline{J}_{\mathrm{C}}(\underline{m})$ is equal to*

$$\mathfrak{w}\left(\overline{J}_{\mathrm{C}}(\underline{m})\right) = (1+t)^{2g(\mathrm{C}^\nu)} t^{2h^1(\Gamma)} c(\Gamma).$$

Proof. This follows from Proposition 3.8 using that $\mathfrak{w}(\mathbb{L}) = t^2$ and that $\mathfrak{w}(J_{\mathrm{C}^\nu}) = (1+t)^{2g^\nu(\mathrm{C})}$ because $\mathfrak{w}(J_{\mathrm{C}^\nu})$ is an abelian variety of dimension $g^\nu(\mathrm{C})$. □

3.4. **Determination of** $IC(\Lambda^i R^1 \pi_{sm*}\overline{\mathbb{Q}}_\ell)$**.** In this section we go back to the setting of §3.1: C is a *nodal* curve defined over a finite field $\mathbb{F}_\pi$ and $\Gamma = \Gamma_{\overline{\mathrm{C}}}$ is the dual graph of $\overline{\mathrm{C}} = \mathrm{C} \otimes_{\mathbb{F}_\pi} \overline{\mathbb{F}_\pi}$. Let $\pi : \mathcal{C} \to B$ be a locally versal family of curves with central fibre the curve $\mathrm{C} = \mathcal{C}_b$ and assume, up to localizing at b, that B is smooth and irreducible.

We are interested in computing the stalk at the special point b of the IC sheaf associated to the local system given by the i'th exterior power of the first cohomology of the curve at the general point. After base-changing by some finite extension of $\mathbb{F}_\pi$, the discriminant locus Δ of π is a normal crossings divisor on B which has a component Δ_e for each node e of $\overline{C}$.

On $B_{\mathrm{reg}} := B\backslash\Delta$ we have the local system $\mathscr{V}^1 := R^1 f_* \overline{\mathbb{Q}}_{\ell|B_{\mathrm{reg}}}$, which, as explained in Appendix A, defines a local system $\mathscr{V}^1_{\bigcap_e \Delta_e}$ on $\bigcap_e \Delta_e \ni b$, endowed with $|E|$ commuting twisted nilpotent endomorphisms

$$N_e : \mathscr{V}^1_{\bigcap_e \Delta_e} \to \mathscr{V}^1_{\bigcap_e \Delta_e} \otimes \mathbb{L}.$$

We also have the local systems $\mathscr{V}^i := \bigwedge^i \mathscr{V}^1$, and corresponding sheaves $\mathscr{V}^i_{\bigcap_e \Delta_e}$ on $\bigcap_e \Delta_e$, endowed with commuting twisted nilpotent endomorphisms

$$N_e^{(i)} : \mathscr{V}^i_{\bigcap_e \Delta_e} \to \mathscr{V}^i_{\bigcap_e \Delta_e} \otimes \mathbb{L}.$$

It is known that the local system $\mathscr{V}^1$, and therefore also its exterior powers $\mathscr{V}^i$, are tamely ramified [Ab, Thm. 1.5]. As we are interested in pointwise computations, we may consider a normal slice so we assume $\bigcap_e \Delta_e = \{b\}$ and identify $\mathscr{V}^1_{\bigcap_e \Delta_e} \cong H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, where $\mathcal{C}_{\overline{\eta}}$ is a one-variable smoothing of C. Remark that the monodromy filtration is independent of the one-variable smoothing that we choose as it coincides with the weight filtration (see Proposition A.2). The monodromy-weight filtration of $\mathscr{V}^1_{\bigcap_e \Delta_e}$ is hence identified with that of $H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ described in §3.1.2.

It follows immediately from weights considerations that the map

$$N_e : H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}$$

factors as

$$H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} = Gr_2^W H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to Gr_0^W H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} = H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}, \tag{21}$$

and it is easily seen to be given by

$$H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \hookrightarrow \mathbb{E} \xrightarrow{t \mapsto \langle \vec{e}^*, t\rangle \cdot \vec{e}^*} \mathbb{E}^* \twoheadrightarrow H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$$

where $\vec{e}$ is an orientation of the edge e and $\vec{e}^*$ is its dual element in $\mathbb{E}^*$ (note that the above is independent of the orientation of e). Similarly, for the exterior powers, we have the identification $\mathscr{V}^i_{\bigcap_e \Delta_e} \cong \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ under which the operators $N_e^{(i)}$ become

$$\begin{aligned} N_e^{(i)} = \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \ &\to \ \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} \\ c_1 \wedge \cdots \wedge c_i \ &\mapsto \ \sum_{k=1}^i c_1 \wedge \cdots \wedge N_e(c_k) \wedge \cdots \wedge c_i. \end{aligned}$$

For $I \subset \mathrm{E}$ we write

$$N_I^{(i)} := \prod_{e \in I} N_e^{(i)} : \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}^{|I|}.$$

As recalled in Appendix A, the stalk of $IC(\mathscr{V}^i)$ at $\{b\} = \bigcap_e \Delta_e$ is quasi-isomorphic to the following complex of continuous $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{Gal}(\overline{k}/k)$:

$$0 \to \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to \bigoplus_{I \subseteq E, |I|=1} \operatorname{Im} N_I^{(i)} \to \bigoplus_{I \subseteq E, |I|=2} \operatorname{Im} N_I^{(i)} \to \cdots \tag{22}$$

We denote this complex by $\mathbf{C}^\bullet(\{N_n\}_{n \in \mathrm{E}}, \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell))$; the first term $\bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$ is in homological degree zero.

We also define operators by restricting the above to the even weight pieces of the associated graded pieces, $H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)_{ev} := H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \oplus H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}$, i.e.,

$$\hat{N}_e : H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \oplus H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} \to H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L} \oplus H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}^2,$$

and similarly for the operators $\hat{N}_e^{(i)}$ and $\hat{N}_I^{(i)}$.

We want now to describe the image of the maps $\hat{N}_I^{(i)}$. Recall from Lemma 3.4 that if $I \in \mathscr{C}(\Gamma)$, then there is an injective map $\wedge \underline{e}_I^* : \bigwedge^{i-|I|} H^1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \to \bigwedge^i H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$. Using the natural injection $H_1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \hookrightarrow H_1(\Gamma, \overline{\mathbb{Q}}_\ell)$ coming from the inclusion of graphs $\Gamma \backslash I \subset \Gamma$, we get an injective map

$$\wedge e_I^* : \bigwedge^{i-|I|} \left(H^1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \oplus H_1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}\right) \hookrightarrow \bigwedge^i \left(H^1(\Gamma, \overline{\mathbb{Q}}_\ell) \oplus H_1(\Gamma, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}\right). \tag{23}$$

Lemma 3.11. (the main calculation) *The image of $\hat{N}_I^{(i)}$ is zero unless $I \in \mathscr{C}(\Gamma)$, and in this case, it is equal to to the image of the map (23) twisted by $\mathbb{L}^{|I|}$.*

Proof. We recall how the choice of a spanning forest of Γ (i.e. a spanning tree on each connected component of Γ) gives rise to dual bases for $H_1(\Gamma) := H_1(\Gamma, \overline{\mathbb{Q}}_\ell)$ and $H^1(\Gamma) := H^1(\Gamma, \overline{\mathbb{Q}}_\ell)$. Let $J \subseteq \mathrm{E}$ be a maximal element of $\mathscr{C}(\Gamma)$ so that $\Gamma \backslash J$ is a spanning forest of Γ. Then on one hand, for each $e \in J$, we have the corresponding $\vec{e}^* \in \mathbb{E}^*$, and their images in $H^1(\Gamma)$ give a basis. On the other hand, for each $e \in J$, there is unique loop in $\Gamma \backslash (J \backslash e)$ which gives rise to an element of $H_1(\Gamma)$ denoted by $\underline{e}$; this again gives a basis. We have $\langle \vec{e}_i^*, \underline{e}_j \rangle = \pm \delta_{ij}$ for each $e_i, e_j \in E$.

We return to the problem at hand. By induction on $|I|$ and the obvious compatibility of N_e with the analogous operator on the complex associated to a subgraph $\Gamma \backslash e'$, it suffices to consider the case when $I = \{e\}$. Let Γ_e be the component of Γ containing e. If the removal of the edge e disconnects Γ_e, then certainly no cycle $t \in H_1(\Gamma)$ can contain the edge e, hence $\langle \vec{e}^*, t \rangle = 0$ for any t, and so $N_e \equiv 0$.

Otherwise, there exists some maximal $e \in J \in \mathscr{C}(\Gamma)$. Let $\{\underline{e} = \underline{e}_1, \underline{e}_2, \ldots\}$ and $\{\vec{e}^* = \vec{e}_1^*, \vec{e}_2^*, \ldots\}$ be the corresponding dual bases. Observe that $J \backslash e \in \mathscr{C}(\Gamma \backslash e)$ is again maximal, and the resulting dual basis of $H_1(\Gamma \backslash e)$ and $H^1(\Gamma \backslash e)$ are $\{\underline{e}_2, \ldots\}$ and $\{\vec{e}_2^*, \ldots\}$.

We compute the action of $\hat{N}_e^{(i)}$:

$$\begin{aligned}
& \hat{N}_e^{(i)}(\vec{e}_{a_1}^* \wedge \cdots \wedge \vec{e}_{a_d}^* \wedge \underline{e}_{b_1} \wedge \cdots \wedge \underline{e}_{b_{i-d}}) \\
= \; & \sum_{r=1}^{i-d} \vec{e}_{a_1}^* \wedge \cdots \wedge \vec{e}_{a_d}^* \wedge \underline{e}_{b_1} \wedge \cdots \wedge \hat{N}_e(\underline{e}_{b_r}) \wedge \cdots \wedge \underline{e}_{b_{i-d}} \\
= \; & \left(\sum_{r=1}^{i-d} \pm \delta_{1,b_r} \cdot \vec{e}_{a_1}^* \wedge \cdots \wedge \vec{e}_{a_d}^* \wedge \underline{e}_{b_1} \wedge \cdots \wedge \hat{\underline{e}}_{b_r} \wedge \cdots \wedge \underline{e}_{b_{i-d}} \right) \wedge \vec{e}^*
\end{aligned}$$

If any of the $a_i = 1$, then this sum vanishes. In any case, the sum has at most one nonvanishing term, that of $b_r = 1$. Assuming without loss of generality that $a_1 < a_2 < \cdots$ and $b_1 < b_2 < \cdots$, the sum vanishes unless $a_1 > 1$ and $b_1 = 1$; and

$$\hat{N}_e^{(i)}(\underline{e}^*_{a_1>1} \wedge \cdots \wedge \underline{e}^*_{a_d} \wedge \underline{e} \wedge \underline{e}_{b_2>1} \wedge \cdots \wedge \underline{e}_{b_{i-d}}) = \pm \underline{e}^* \wedge \left(\underline{e}^*_{a_1>1} \wedge \cdots \wedge \underline{e}^*_{a_d} \wedge \underline{e}_{b_2>1} \wedge \cdots \wedge \underline{e}_{b_{i-d}}\right)$$

This completes the proof. $\square$

Remark 3.12. In particular, if $i < |I|$ or $h^1(\Gamma) < |I|$ then $\hat{N}_I^{(i)}$ vanishes. This is true also for the map

$$N_I^{(i)} : \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \to \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}^{|I|}.$$

Indeed, if $i < |I|$ then $N_I^{(i)}$ vanishes because the weights of the source go from 0 to $2i$ while those of the target from $2|I|$ to $2|I| + 2i$. Moreover, if $h^1(\Gamma) < |I|$ then the map $N_I^{(i)}$ vanishes because of the factorization (21).

We now prove the main result of this subsection.

Theorem 3.13. *The class of* $\sum q^i IC\left(\bigwedge^i R^1 f_* \overline{\mathbb{Q}}_{\ell|B_{\rm reg}}\right)_b [-i]$ *in* $K_0(\mathrm{Rep}(\mathrm{Gal}(\overline{k}/k), \overline{\mathbb{Q}}_\ell))[[q]]$ *is given by the formula*

(24)
$$Z_{\overline{\mathrm{C}}^\nu} \cdot \sum_{I \in \mathscr{C}(\Gamma)} \left(\left(\prod_{e \in I} q\mathbb{L}\chi_e \right) \left(\prod_{e \notin I} (1 - q\chi_e)(1 - q\mathbb{L}\chi_e) \right) \left(\prod_{c \in \pi_0(\Gamma)} (1 - q \cdot [c])(1 - q\mathbb{L} \cdot [c]) \right) \right),$$

where $Z_{\overline{\mathrm{C}}^\nu}$ *is the cohomological Zeta function of* $\overline{\mathrm{C}}^\nu$*:*

$$Z_{\overline{\mathrm{C}}^\nu} := \sum_{n=1}^{\infty} q^n \sum_i (-1)^i H^i((\overline{\mathrm{C}}^\nu)^{(n)}, \overline{\mathbb{Q}}_\ell) = \frac{\sum_i (-q)^i \bigwedge^i H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)}{\left(\sum_i (-q)^i \bigwedge^i H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)\right)\left(\sum_i (-q)^i \bigwedge^i H^2(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)\right)}.$$

Proof. As discussed above, the "CKS complex" calculation of the stalk gives the formula

$$\sum_i \left(q^i IC\left(\bigwedge^i R^1 f_* \overline{\mathbb{Q}}_{\ell|B_{\rm reg}}\right)_b [-i]\right) = \sum_i \left(q^i \mathbf{C}^\bullet(\{N_e\}_{e \in E}, \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell))[-i]\right).$$

We split the right hand side into contributions from $\overline{C}^\nu$ and the pieces on which there is actually monodromy. That is,

$$\sum \left(q^i \mathbf{C}^\bullet(\{N_e\}_{e \in E}, \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell))[-i]\right)$$

is equal to

$$\left(\sum_i (-q)^i \bigwedge^i H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)\right)\left(\sum_i (-q)^i \mathbf{C}^\bullet(\{\hat{N}_e\}_{e \in E}, \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)_{ev})\right).$$

We have

$$\begin{aligned} \mathbf{C}^k(\{\hat{N}_e\}_{e \in E}, \bigwedge^i H^1(\mathcal{C}_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)_{ev}) &:= \bigoplus_{I \in \mathscr{C}_k(\Gamma)} \mathrm{Im}\, \hat{N}_I^{(i)} \\ &= \mathbb{L}^k \otimes \left(\bigoplus_{I \in \mathscr{C}_k(\Gamma)} e_I^* \wedge \bigwedge^{i-|I|} \left(H^1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \oplus H_1(\Gamma \backslash I, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{L}\right)\right) \end{aligned}$$

Were the first equality is definitional and the second is the result of Lemma 3.11,

Since the Galois action on the $\wedge e_I^*$ is the same as on $\prod_{e\in I}\chi_e$, passing to $K_0(\mathrm{Rep}(\mathrm{Gal}(\overline{k}/k),\overline{\mathbb{Q}}_\ell))$ we have

$$\begin{aligned}\mathbf{C}^k(\{\hat{N}_e\}_{e\in E},\textstyle\bigwedge^i H^1(\mathcal{C}_{\overline{\eta}},\overline{\mathbb{Q}}_\ell)_{ev}) &= \mathbb{L}^k\cdot\sum_{I\in\mathscr{C}_k(\Gamma)}\left(\prod_{e\in I}\chi_e\right)\cdot\textstyle\bigwedge^{i-|I|}\left(H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)+H_1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)\cdot\mathbb{L}\right)\\ &= \mathbb{L}^k\cdot\sum_{I\in\mathscr{C}_k(\Gamma)}\left(\prod_{e\in I}\chi_e\right)\cdot\textstyle\bigwedge^{i-|I|}\left(H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)\cdot(1+\mathbb{L})\right).\end{aligned}$$

The final equality holds since $H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)$ and $H_1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)$ are isomorphic as Galois representations; this can be seen e.g. because the Galois group acts as signed permutations on $\mathbb{E},\mathbb{V}$. Summing, we have

$$\begin{aligned}&\sum_i(-q)^i\mathbf{C}^\bullet(\{\hat{N}_e\}_{e\in E},\bigwedge^i H^1(\mathcal{C}_{\overline{\eta}},\overline{\mathbb{Q}}_\ell)_{ev})\\ =\ &\sum_i(-q)^i\left(\sum_{I\in\mathscr{C}(\Gamma)}(-\mathbb{L})^{|I|}\cdot\left(\prod_{e\in I}\chi_e\right)\cdot\textstyle\bigwedge^{i-|I|}\left(H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)\cdot(1+\mathbb{L})\right)\right)\\ =\ &\sum_{I\in\mathscr{C}(\Gamma)}(q\mathbb{L})^{|I|}\left(\prod_{e\in I}\chi_e\right)\cdot\sum_j(-q)^j\textstyle\bigwedge^j\left(H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)\cdot(1+\mathbb{L})\right).\end{aligned}$$

We wish to write things in terms of $\mathbb{E}$ rather than $H^1(\Gamma,\overline{\mathbb{Q}}_\ell)$. Because

$$H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)+\mathbb{V}^*_{\Gamma\backslash I}=\mathbb{E}^*_{\Gamma\backslash I}+H^0(\Gamma,\overline{\mathbb{Q}}_\ell) \tag{25}$$

and moreover, $\mathbb{V}_\Gamma=\mathbb{V}_{\Gamma\backslash I}=\mathbb{V}^*_{\Gamma\backslash I}$, we have:

$$\begin{aligned}&\left(\sum_i(-q)^i\textstyle\bigwedge^i(\mathbb{V}_\Gamma\cdot(1+\mathbb{L}))\right)\left(\displaystyle\sum_i(-q)^i\mathbf{C}^\bullet(\{\hat{N}_e\}_{e\in E},\textstyle\bigwedge^i H^1(\mathcal{C}_{\overline{\eta}},\overline{\mathbb{Q}}_\ell)_{ev})\right)\\ =\ &\left(\sum_i(-q)^i\textstyle\bigwedge^i(\mathbb{V}_\Gamma\cdot(1+\mathbb{L}))\right)\left(\displaystyle\sum_{I\in\mathscr{C}(\Gamma)}\left(\prod_{e\in I}q\mathbb{L}\chi_e\right)\cdot\sum_j(-q)^j\textstyle\bigwedge^j\left(H^1(\Gamma\backslash I,\overline{\mathbb{Q}}_\ell)\cdot(1+\mathbb{L})\right)\right)\\ =\ &\sum_{I\in\mathscr{C}(\Gamma)}\left(\prod_{e\in I}q\mathbb{L}\chi_e\right)\cdot\sum_j(-q)^j\textstyle\bigwedge^j\left((\mathbb{E}^*_{\Gamma\backslash I}+H^0(\Gamma,\overline{\mathbb{Q}}_\ell))\cdot(1+\mathbb{L})\right)\\ =\ &\sum_{I\in\mathscr{C}(\Gamma)}\left(\prod_{e\in I}q\mathbb{L}\chi_e\right)\left(\prod_{e\notin I}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\left(\prod_{c\in\pi_0(\Gamma)}(1-q\cdot[c])(1-q\mathbb{L}\cdot[c])\right).\end{aligned}$$

Note finally that $\mathbb{V}_\Gamma = H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)$. Thus the extra term we introduced fits naturally with the first term we factored out:

$$\begin{aligned}\frac{\sum_i (-q)^i \bigwedge^i H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)}{\sum_i (-q)^i \bigwedge^i (\mathbb{V}_\Gamma \cdot (1+\mathbb{L}))} &= \frac{\sum_i (-q)^i \bigwedge^i H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)}{\sum_i (-q)^i \bigwedge^i \left(H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell) \cdot (1+\mathbb{L})\right)} \\ &= \frac{\sum_i (-q)^i \bigwedge^i H^1(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)}{\left(\sum_i (-q)^i \bigwedge^i H^0(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)\right)\left(\sum_i (-q)^i \bigwedge^i H^2(\overline{\mathrm{C}}^\nu, \overline{\mathbb{Q}}_\ell)\right)}.\end{aligned}$$

That is, this fraction is the (cohomological) Zeta function of $\overline{\mathrm{C}}^\nu$, according to MacDonald's formula [Mac] for the smooth curve $\overline{\mathrm{C}}^\nu$. □

Remark 3.14. To read the above proof, it may be helpful to recall $\sum_i (-q)^i \bigwedge^i(X) = Sym^*(-qX)$; we have eschewed this for the benefit of the reader not accustomed to λ-ring manipulations.

Using the above Theorem, we can compute the weight polynomial of the term appearing in (1). We will use the notation of Definition 3.1.

Corollary 3.15. *We have the following evaluations of weight polynomials:*

$$\mathfrak{w}\left(\sum_i q^i IC\left(\bigwedge^i R^1 f_* \overline{\mathbb{Q}}_{\ell|B_{\mathrm{reg}}}\right)_b [-i]\right) = (1+qt)^{2g(\overline{\mathrm{C}}^\nu)} \sum_{i \geqslant 0} n_i(\Gamma) \cdot (qt^2)^{h^1(\Gamma)-i} \left((1-qt^2)(1-q)\right)^i.$$

In particular, we get that

$$\mathfrak{w}\left(\sum_i IC\left(\bigwedge^i R^1 f_* \overline{\mathbb{Q}}_{\ell|B_{\mathrm{reg}}}\right)_b [-i]\right) = (1+t)^{2g(\overline{\mathrm{C}}^\nu)} t^{2h^1(\Gamma)} c(\Gamma).$$

Proof. In order to prove the first statement, we will compute the weight polynomial of (24). The weight polynomial of $Z_{\overline{\mathrm{C}}^\nu}$ is obtained by replacing the pure weight i space $H^i(\overline{C}^\nu, \overline{\mathbb{Q}}_\ell)$ by $(-1)^i t$ times its dimension; hence we get

$$\mathfrak{w}(Z_{\overline{\mathrm{C}}^\nu}) = \frac{(1+qt)^{2g(\overline{\mathrm{C}}^\nu)}}{(1-q)^{|\mathbb{V}_\Gamma|}(1-qt^2)^{|\mathbb{V}_\Gamma|}}. \tag{26}$$

The weight polynomial of the second term in (24) is obtained by setting $\mathbb{L} = t^2$, $\chi_e = [c] = 1$ (because the weight polynomial discards the Galois action) and noticing that if $I \in \mathscr{C}(\Gamma)$ then by definition we have that $h^0(\Gamma) = h^0(\Gamma \backslash I)$ and $h^1(\Gamma) - h^1(\Gamma\backslash I) = |I|$; hence we get

$$\begin{aligned}&\mathfrak{w}\left(\sum_{I \in \mathscr{C}(\Gamma)} \left(\prod_{e \in I} q\mathbb{L}\chi_e\right)\left(\prod_{e \notin I}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\left(\prod_{c \in \pi_0(\Gamma)}(1-q\cdot[c])(1-q\mathbb{L}\cdot[c])\right)\right) = \\ &= \sum_{I \in \mathscr{C}(\Gamma)} \left(qt^2\right)^{h^1(\Gamma)-h^1(\Gamma\backslash I)} \left((1-q)(1-qt^2)\right)^{|\mathbb{E}_{\Gamma\backslash I}|+h^0(\Gamma)} = \\ &= \sum_{i=0}^{h^1(\Gamma)} n_i(\Gamma) \left(qt^2\right)^{h^1(\Gamma)-i} \left((1-q)(1-qt^2)\right)^{i+|\mathbb{V}_\Gamma|}, \end{aligned} \tag{27}$$

where in the last equation we have used (25) and the Definition 3.1 of $n_i(\Gamma)$. By multiplying (26) and (27), we get the first statement.

For the second statement, we set $q = 1$ in the first statement, which forces all terms in the sum to vanish except those for which $\dim H_1(\Gamma\backslash I) = 0$ and then use that $n_0(\Gamma) = c(\Gamma)$. □

4. Relative compactified Jacobian for non-versal families

The main result of this section, namely Theorem 4.11, gives sufficient conditions for a relative fine compactified Jacobian of a non-versal family to be nonsingular. If a family of curves $\mathcal{C} \to S$ contains only irreducible curves, then the relative compactified Jacobian is non singular if and only if the relative Hilbert schemes of any length are non singular [Sh]. The if implication is still true for families of reducible curves (as we will show in Corollary 4.16), but the only if implication is no longer true: already in arithmetic genus one, banana curve, or a triangle of lines, give examples of fine compactified Jacobians which can be smoothed in a one-dimensional family, whereas the Hilbert scheme of length two of the curve needs at least a two-dimensional family. It should be clear from the proof of Theorem 4.11 that the reason for this discrepancy is that certain torsion free sheaves, which, as points of the Hilbert scheme, can be smoothed only in a high dimensional family, cannot appear in the compactified Jacobian because of the stability condition. For instance, in the triangle, a torsion-free sheaf is contained in a fine compactified Jacobian if and only if it is locally free outside at most one point.

The proof of this fact, of independent interest, is based on the results of [FGvS] and a local duality theorem due to T. Warmt [W], which we now review. All the unproven facts here may be found in [W, Chapter 4] and [FGvS].

Fix the following data:

(1) a planar complete local ring $R = k[[x,y]]/(f)$, with $f = \prod_{a\in\Lambda} f_a$ and $f_a \in k[[x,y]]$ irreducible elements; assume that k is an algebraically closed field of arbitrary characteristic. The set Λ is the set of branches of R, i.e. minimal prime ideals of R, and we set $\lambda := \sharp\Lambda$. The normalization $\tilde{R}$ of R is isomorphic to $\tilde{R} \simeq \prod_{a\in\Lambda} k[[T_a]]$, where T_a is a parameter on the a-th branch. Observe that $\tilde{R}$ contains R and it is a subring of the total fraction field $Q(R) \simeq \prod_{a\in\Lambda} k((T_a))$.

(2) a rank one, torsion-free R-module M, which, up to isomorphism, we can assume to contain R and to be contained in $\widetilde{R}$:
$$R \subseteq M \subseteq \widetilde{R}.$$

Consider the conductor ideal of the extension $\tilde{R}/R$
$$\mathfrak{f} = \mathrm{Ann}(\widetilde{R}/R) = \mathrm{Hom}_R(\widetilde{R}, R) = \{u \in R \text{ such that } u\widetilde{R} \subset R\},$$
which is the biggest ideal of $\tilde{R}$ contained in R. The delta-invariant of the ring R is defined as $\delta(R) := \dim \widetilde{R}/R$. Since R is Gorenstein by our assumptions, we have that
$$\delta(R) = \dim R/\mathfrak{f} = \frac{1}{2}\dim \widetilde{R}/\mathfrak{f}. \tag{28}$$

One can associate to the module M two objects of primary importance:

- The first Fitting ideal $\mathrm{Fit}_1(M)$ of M, defined as the ideal generated by $(N-1)$-minors of a free resolution
$$0 \longleftarrow M \longleftarrow k[[x,y]]^N \longleftarrow k[[x,y]]^N \longleftarrow 0$$
of M as a $k[[x,y]]$-module. Under the hypotheses above, we have that
$$\mathrm{Fit}_1(M) = \{\phi(m), \text{ for } m \in M \text{ and } \phi \in \mathrm{Hom}_R(M,R)\},$$
and $\mathfrak{f} \subseteq \mathrm{Fit}_1(M)$, see [FGvS, Prop. C-2 and Cor. C-3].

- The endomorphism ring of M
$$\mathrm{End}_R(M) = \{c \in \widetilde{R} \, : \, cm \in M \text{ for all } m \in M\}.$$
which is a subring of $\widetilde{R}$ containing R and contained in M. Notice that $\mathrm{End}_R(M)$ may not be planar and not even Gorenstein.

We have the series of inclusions
$$\mathfrak{f} \subseteq \mathrm{Fit}_1(M) \subseteq R \subseteq \mathrm{End}_R(M) \subseteq M \subseteq \widetilde{R}.$$
The first Fitting ideal of M is dual to the endomorphism ring of M, as stated in the following result.

Proposition 4.1. *(*[W, Korollar 4.4.2, ii]*) Under the hypotheses above, the map*
$$\begin{aligned} \mathrm{Hom}_R(\mathrm{End}_R(M), R) &\longrightarrow \mathrm{Fit}_1(M) \\ \Psi &\mapsto \Psi(\mathrm{id}) \end{aligned}$$
is an isomorphism.

Using the endomorphism ring of a module M, we can introduce an important numerical invariant of M.

Definition 4.2. Let $\nu = (\lambda_1, \ldots, \lambda_{l(\nu)})$ be a partition of $\lambda = \sharp \Lambda$. We say that M has type ν if $\mathrm{End}_R(M)$ is direct product of $l(\nu)$ local rings, the i-th of which has λ_i branches. The type of M is denoted by $\nu(M)$.

Given a partition $\nu = (\lambda_1, \ldots, \lambda_{l(\nu)})$ as above, let
$$I_1 = \{1, \cdots, \lambda_1\}, I_2 = \{\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2\}, \cdots I_{l(\nu)} = \{\lambda - \lambda_{l(\nu)} + 1, \cdots, \lambda\}$$
and let R_ν be the subring of $\tilde{R}$ given by
$$R_\nu = \{(f_1(T_1), \cdots, f_\lambda(T_\lambda)) \in \prod_{i=1}^{\lambda} k[[T_i]] \text{ with } f_k(0) = f_l(0) \text{ if } k, l \in I_j \text{ for some } j\}.$$
Geometrically, R_ν is the disjoint union of the complete local rings at 0 of the coordinate axes in $\mathbb{A}^{n_i}$, for $i = 1, \ldots, l(\nu)$. Therefore, the rings R_ν are seminormal and, indeed, they are all the seminormal rings containing R and contained in $\widetilde{R}$. If a partition ν' refines ν then we have that $R_\nu \subseteq R_{\nu'}$; the two extreme case being $R_{(1,\ldots,1)} = \tilde{R}$ and $R_{(\lambda,0,\ldots,0)}$ which is the seminormalization of R. The delta invariant of R_ν is easily seen to be equal to
$$\dim \tilde{R}/R_\nu := \delta(R_\nu) = \lambda - l(\nu). \tag{29}$$
From Proposition 4.1 and using that $\mathfrak{f} = \mathrm{Fit}_1(\widetilde{R})$, we deduce that
$$\dim \mathrm{Fit}_1(R_\nu)/\mathfrak{f} = \lambda - l(\nu). \tag{30}$$
From [Liu, Chapter 7, Ex. 5.9], we deduce the following alternative characterization of the type of M.

Lemma 4.3. *The type of M is the coarsest partition ν such that $\mathrm{End}_R(M) \subseteq R_\nu$. Hence, $R_{\nu(M)}$ is the seminormalization of $\mathrm{End}_R(M)$.*

From the above characterization of the type of M and Proposition 4.1, we deduce the following:

Corollary 4.4. *For any module as above, we have that $\mathrm{Fit}_1(M) \supseteq \mathrm{Fit}_1(R_{\nu(M)})$.*

We now review the nonsingularity condition for a relative fine compactified Jacobian at a given point: the reference is again [FGvS]. A clear recollection of the results can be found in [W, §4.5].

Let C be a projective reduced connected curve with planar singularities over $k = \overline{k}$, $\mathrm{C}_{\mathrm{sing}} = \{c_1, \cdots, c_r\}$ its singular set, $\{\Lambda_1, \cdots, \Lambda_r\}$ the corresponding sets of branches, with cardinality $\lambda_i := \sharp\Lambda_i$.

Given a singular point $c_i \in \mathrm{C}_{\mathrm{sing}}$, let f_i be a local equation of C at c_i, so that $\hat{\mathcal{O}}_{\mathrm{C},c_i} \simeq k[[x,y]]/(f_i)$. We have the semiuniversal deformation space $\mathbb{V}_i := \mathrm{Def}(\mathrm{C}, c_i)$ of the local ring $\hat{\mathcal{O}}_{\mathrm{C},c_i}$, whose tangent space $T\mathbb{V}_i$ is the underlying vector space of the k-algebra $k[[x,y]]/(f_i, \partial_x f_i, \partial_y f_i)$. There is the canonical subspace $\mathbb{V}_i^\delta \subset T\mathbb{V}_i$, the support of the tangent cone at c_i of the equigeneric locus. The subspace $\mathbb{V}_i^\delta$ is the class in $T\mathbb{V}_i = k[[x,y]]/(f_i, \partial_x f_i, \partial_y f_i)$ of the conductor ideal $\mathfrak{f}_i := \mathrm{Ann}(\widetilde{\mathcal{O}}_{\mathrm{C},c_i}/\mathcal{O}_{\mathrm{C},c_i})$. By (28), we have that

$$\operatorname{codim} \mathbb{V}_i^\delta = \delta(c_i) = \dim_k \widetilde{\mathcal{O}}_{\mathrm{C},c_i}/\mathcal{O}_{\mathrm{C},c_i}. \tag{31}$$

Given a partition ν_i of the set Λ_i of branches at c_i, we have the partial normalization with local ring $(\mathcal{O}_{\mathrm{C},c_i})_{\nu_i}$ and the subspace $\mathbb{V}_i^{\nu_i}$, representing the class in $T\mathbb{V}_i = k[[x,y]]/(f_i, \partial_x f_i, \partial_y f_i)$ of the ideal $\mathrm{Fit}_1((\mathcal{O}_{\mathrm{C},c_i})_\nu)$. By (30), we have that

$$\dim \mathbb{V}_i^{\nu_i}/\mathbb{V}_i^\delta = \dim \mathrm{Fit}_1((\mathcal{O}_{\mathrm{C},c_i})_\nu)/\mathfrak{f}_i = \lambda_i - l(\nu_i). \tag{32}$$

We set $\mathbb{V} := \mathrm{Def}^{\mathrm{loc}}(\mathrm{C}) = \prod \mathbb{V}_i$ and $\mathbb{V}^\delta := \prod \mathbb{V}_i^\delta \subset T\mathbb{V} = \prod T\mathbb{V}_i$, a codimension $\delta(\mathrm{C}) = \sum \delta(c_i)$ linear subspace. Given a multipartition $\underline{\nu} = \{\nu_i\}$, where ν_i is a partition of λ_i, we have the subspace $\mathbb{V}^{\underline{\nu}} := \prod \mathbb{V}_i^{\nu_i} \subset T\mathbb{V}$ and the corresponding partial normalization $\mathrm{C}^{\underline{\nu}}$ of C, with local ring $(\mathcal{O}_{\mathrm{C},c_i})_{\nu_i}$ at the point $c_i \in \mathrm{C}$. The curve $C^{\underline{\nu}}$ is seminormal and indeed all seminormal partial normalizations of C are of the form $\mathrm{C}^{\underline{\nu}}$ for some unique multipartition $\underline{\nu} = \{\nu_i\}$. By (32), we get that

$$\operatorname{codim} \mathbb{V}^{\underline{\nu}} = \sum_{i=1}^r \operatorname{codim} \mathbb{V}_i^{\nu_i} = \sum_{i=1}^r (\delta(c_i) + l(\nu_i) - \lambda_i) = \delta(\mathrm{C}) + \sum_{i=1}^r (l(\nu_i) - \lambda_i). \tag{33}$$

Let $\mathcal{I}$ be a rank one torsion free sheaf on C with stalk $\mathcal{I}_i$ at c_i. The local semiuniversal deformation space $\mathrm{Def}((\mathrm{C}, c_i), \mathcal{I}_i)$ of the pair $(\hat{\mathcal{O}}_{\mathrm{C},c_i}, \mathcal{I}_i)$ is endowed with a forgetful morphism $\rho_i : \mathrm{Def}((\mathrm{C}, c_i), \mathcal{I}_i) \to \mathbb{V}_i = \mathrm{Def}(\mathrm{C}, c_i)$ and we set

$$\rho := \prod \rho_i : \prod \mathrm{Def}((\mathrm{C}, c_i), \mathcal{I}_i) \to \mathbb{V}.$$

Let $W_i(\mathcal{I}) = \mathrm{Im}(d\rho_i)$ and $W(\mathcal{I}) = \mathrm{Im}(d\rho) = \prod W_i(\mathcal{I})$ be the images of the differentials.

The linear subspace $W_i(\mathcal{I})$ is determined by the first Fitting ideal of $\mathcal{I}_i$.

Proposition 4.5. *(*[FGvS, Prop. C-1]*) The subspace $W_i(\mathcal{I})$ is the class in $k[[x,y]]/(f_i, \partial_x f_i, \partial_y f_i)$ of the first Fitting ideal $\mathrm{Fit}_1(\mathcal{I}_i)$ of the stalk $\mathcal{I}_i$ of $\mathcal{I}$ at c_i.*

The linear subspace $W(\mathcal{I})$ allows to characterize when a relative fine compactified Jacobian is regular at the point $\mathcal{I}$. Given a family of curves $\pi : \mathcal{C} \to S$ and a point $o \in S$ such that $\mathrm{C} := \pi^{-1}(o) = \mathcal{C}_o$, let $d\sigma : T_o S \to T\mathbb{V} = \prod T\mathbb{V}_i$ be the differential of the associated map to the product of the first-order deformations of the singularities.

Proposition 4.6. *(*[FGvS, Cor. B-3]*) Given a family $\pi : \mathcal{C} \to S$, with $\mathrm{C} = \mathcal{C}_o$, a relative fine compactified Jacobian $\overline{J}_{\mathcal{C}}$ is regular at a point $\mathcal{I}$ lying in the central fiber $(\overline{J}_{\mathcal{C}})_o = \overline{J}_{\mathrm{C}}$ if and only if $W(\mathcal{I}) + \mathrm{Im}(\mathrm{d}\sigma) = \mathrm{T}\mathbb{V}$.*

Analogously to 3.3, consider the endomorphism sheaf $\underline{\mathrm{End}}_{\mathcal{O}_{\mathrm{C}}}(\mathcal{I})$ of $\mathcal{I}$: it is a sheaf of finite $\mathcal{O}_{\mathrm{C}}$-algebras such that $\mathcal{O}_{\mathrm{C}} \subseteq \underline{\mathrm{End}}_{\mathcal{O}_{\mathrm{C}}}(\mathcal{I}) \subseteq \mathcal{O}_{\mathrm{C}^{\nu}}$. The sheaf $\mathcal{I}$ is naturally a sheaf on the partial normalization $\mathrm{C}^{\mathcal{I}} := \underline{\mathrm{Spec}}_{\mathrm{C}}(\mathrm{End}_{\mathcal{O}_{\mathrm{C}}}(\mathcal{I}))$ of C; the original $\mathcal{I}$ being recovered by the pushforward along the partial normalization morphism $\nu_{\mathcal{I}} : \mathrm{C}^{\mathcal{I}} \to \mathrm{C}$. For every singular point c_i of C, denote by $\nu_i(\mathcal{I}_i)$ the type of $\mathcal{I}_i$ at c_i (see Definition 4.2) and we set $\underline{\nu}(\mathcal{I}) = \{\nu_i(\mathcal{I}_i)\}$. It follows from Lemma 4.3 that $\mathrm{C}^{\nu(\mathcal{I})}$ is the seminormalization of $\mathrm{C}^{\mathcal{I}}$. The following remark is obvious and it is recorded for later use.

Remark 4.7. The sheaf $\mathcal{I}$ is simple if and only if $\mathrm{C}^{\mathcal{I}}$ is connected, or equivalently, if and only $C^{\underline{\nu}(\mathcal{I})}$ is connected. In particular, if $\mathcal{I}$ belongs to some fine compactified Jacobian of C, then $C^{\underline{\nu}(\mathcal{I})}$ are connected.

We want now to establish a necessary combinatorial criterion in order to check when the partial normalization $C^{\underline{\nu}}$ is connected.

To any reduced projective curve C (not necessarily locally planar), we associate an *hypergraph* $H_{\mathrm{C}} = (V(H_{\mathrm{C}}), E(H_{\mathrm{C}}))$ as follows: the vertices $V(H_C)$ correspond to the irreducible components of C and to each singular point $n \in \mathrm{C}_{\mathrm{sing}}$ we associate an hyperedge e_n which is a multiset of $V(H_C)$ consisting of all irreducible components that contain n, each one of which counted with multiplicity equal to its number of branches at n. In this way, the cardinality $|e_n|$ of the hyperedge e_n is equal to the total number of branches of C at n. Note that if C is a nodal curve, then the hypergraph H_{C} is actually a graph and it coincides with the dual graph of C.

Lemma 4.8. *If the curve* C *is connected then*

$$b(H_C) := \sum_{e \in E(H_C)} (|e| - 1) - |V(H_{\mathrm{C}})| + 1 \geqslant 0.$$

Proof. Clearly the curve C is connected if and only if its associated hypergraph H_{C} is connected, i.e. there does not exist a partition of the vertex set $V(H_C) = V_1 \coprod V_2$ such that every hyperedge e contains only elements of either V_1 or V_2. We will therefore prove more generally that if a hypergraph $H = (V(H), E(H))$ is connected then $b(H) \geqslant 0$.

In order to show this, consider the bipartite simple incidence graph Γ_H constructed from H as follows: its vertices $V(\Gamma_H)$ are the disjoint union of $V(H)$ and of $E(H)$ and its edges are given by $E(\Gamma_H) := \{(v, e) \in V(H) \coprod E(H) : v \in e\}$. Clearly H is connected if and only if Γ_H is connected and, by construction, we have that $|V(\Gamma_H)| = |V(H)| + |E(H)|$ and $|E(\Gamma_H)| = \sum_{e \in E(H)} |e|$. Therefore, if H is connected then $b(H)$ coincides with the first Betti number of $b_1(\Gamma_H) = |E(\Gamma_H)| - |V(\Gamma_H)| + 1$ of Γ_H, which is non-negative. $\square$

We are now ready to prove the main result of this section, namely a sufficient criteria for the regularity of relative fine compactified Jacobians. The criteria will be expressed in terms of the following closed subset of $T\mathbb{V}$:

Definition 4.9. Let C be a curve as above. Consider the closed locus $\mathbb{W} \subset T\mathbb{V}$ given by the union of the linear subspaces $\mathbb{V}^{\underline{\nu}}$, as $\underline{\nu}$ varies among all the maximal multipartitions such that $C^{\underline{\nu}}$ is connected.

The locus $\mathbb{W}$ has the following properties:

Lemma 4.10.

(i) The locus $\mathbb{W} \subset T\mathbb{V}$ *has pure codimension* $\delta^a(\mathrm{C})$.
(ii) We have the inclusion $\mathbb{V}^{\delta} \subseteq \mathbb{W}$ *with equality if and only if* C *is irreducible.*

(iii) *If* $\mathrm{C}^{\underline{\nu}}$ *is connected then* $\mathbb{V}^{\underline{\nu}}$ *contains some irreducible component of* $\mathbb{W}$.

(iv) *If* $\mathcal{I}$ *is a simple torsion-free rank one sheaf then* $W(\mathcal{I})$ *contains some irreducible component of* $\mathbb{W}$.

Proof. Part (i): the irreducible components of $\mathbb{W}$ are given by $\mathbb{V}^{\underline{\nu}}$, where $\underline{\nu}$ is a maximal multipartition such that $\mathrm{C}^{\underline{\nu}}$ is connected. Lemma 4.8 implies that $b(H_{\mathrm{C}^{\underline{\nu}}}) \geqslant 0$. However, due to the maximality of $\underline{\nu}$ we must have that $b(H_{\mathrm{C}^{\underline{\nu}}}) = 0$, for otherwise it is easy to check that we could find a refinement $\underline{\nu}'$ with $\mathrm{C}^{\underline{\nu}'}$ still connected, violating the maximality of $\underline{\nu}$. (This argument is the analogue for a hypergraph of the fact that every connected graph has a spanning tree.) By the definition of $\mathrm{C}^{\underline{\nu}}$, it follows that if $\underline{\nu} = \{\nu_i\}$ with $\nu_i = ((\nu_i)_1, \ldots, (\nu_i)_{l(\nu_i)})$ a partition of the set Λ_i of branches of C at c_i, then the hyperedges of $H_{\mathrm{C}^{\underline{\nu}}}$ have cardinality $(\nu_i)_j$. Therefore, by the definition of $b(H_{\mathrm{C}^{\underline{\nu}}})$, we get

$$0 = b(H_{\mathrm{C}^{\underline{\nu}}}) = \sum_{i=1}^{r}\sum_{k=1}^{l(\nu_i)}((\nu_i)_k - 1) - |V(H_{\mathrm{C}^{\underline{\nu}}})| + 1 = \sum_{i=1}^{r}(\lambda_i - l(\nu_i)) - \gamma(\mathrm{C}) + 1. \tag{34}$$

Combining (33) and (34), we deduce that

$$\operatorname{codim} \mathbb{V}^{\underline{\nu}} = \delta(\mathrm{C}) - \gamma(\mathrm{C}) + 1 = \delta^a(\mathrm{C}),$$

which concludes the proof of part (i).

Part (ii): consider the maximal multipartion $\underline{\nu}_{\max}$, i.e. the one for which each partition ν_i appearing in it has the form $\nu_i = (1, \ldots, 1)$. From the above discussion, it follows that $\mathrm{C}^{\underline{\nu}_{\max}}$ is the normalization $\widetilde{\mathrm{C}}$ of C and that $\mathbb{V}^{\delta} = \mathbb{V}^{\underline{\nu}_{\max}}$. Therefore, we have the inclusion $\mathbb{V}^{\delta} \subseteq \mathbb{W}$ by Proposition 4.1 and Proposition 4.5, and equality holds if and only if $\widetilde{\mathrm{C}}$ is connected, which holds if and only if C is irreducible.

Part (iii): if $\underline{\nu}$ is a multipartition such that $\mathrm{C}^{\underline{\nu}}$ is connected, then as observed above we can find a refinement $\underline{\nu}'$ of $\underline{\nu}$ such that $C^{\underline{\nu}'}$ is connected and it is maximal with this property. Therefore $\mathbb{V}^{\underline{\nu}'}$ is an irreducible component of $\mathbb{W}$ and $\mathbb{V}^{\underline{\nu}} \subseteq \mathbb{V}^{\underline{\nu}'}$ by Proposition 4.1, q.e.d.

Part (iv): if $\mathcal{I}$ is a simple torsion-free rank one sheaf then the seminormalization $\mathrm{C}^{\underline{\nu}(\mathcal{I})}$ of $\mathrm{C}^{\mathcal{I}}$ is connected (see Remark 4.7) and we have that $\mathbb{V}^{\underline{\nu}(\mathcal{I})} \subseteq W(\mathcal{I})$ by Corollary 4.4 and Proposition 4.5. Therefore, we conclude by part (iii). $\square$

Finally we can state and prove the main result of this section.

Theorem 4.11. *Let* $\pi : \mathcal{C} \to S$ *be a flat and projective family of connected curves, with* $\mathrm{C} = \mathcal{C}_o$ *having planar singularities, and let* $d\sigma : T_oS \to T\mathbb{V} = T\,\mathrm{Def}^{\mathrm{loc}}(\mathrm{C})$ *be the differential of the associated map to the product of the first-order deformations of the singularities of* C. *Then a relative fine compactified Jacobian* $\overline{J}_{\mathcal{C}}$ *is regular along* $(\overline{J}_{\mathcal{C}})_o = \overline{J}_{\mathrm{C}}$ *if* $\mathrm{Im}(\mathrm{d}\sigma)$ *is transverse to each irreducible component of* $\mathbb{W}$. *In particular, this is the case if* $\mathrm{Im}(\mathrm{d}\sigma)$ *is a generic subspace of* $\mathbb{V}$ *of dimension at least* $\delta^a(\mathrm{C})$.

Proof. By Proposition 4.6, a relative fine compactified Jacobian $\overline{J}_{\mathcal{C}}$ is regular along $(\overline{J}_{\mathcal{C}})_o = \overline{J}_{\mathrm{C}}$ if and only if $\mathrm{Im}(\mathrm{d}\sigma)$ is transverse to any linear subspace $W(\mathcal{I})$ for any sheaf $\mathcal{I} \in \overline{J}_{\mathrm{C}}$. By Remark 4.7 and Lemma 4.10(iv), any such linear subspace $W(\mathcal{I})$ contains an irreducible component of $\mathbb{W}$; therefore, if $\mathrm{Im}(\mathrm{d}\sigma)$ is transverse to each irreducible component of $\mathbb{W}$, then $\mathrm{Im}(\mathrm{d}\sigma)$ is transverse to every such linear subspace $W(\mathcal{I})$ and the regularity of $\overline{J}_{\mathcal{C}}$ along $(\overline{J}_{\mathcal{C}})_o = \overline{J}_{\mathrm{C}}$ follows.

Since $\mathbb{W}$ has pure codimension $\delta^a(\mathrm{C})$ by Lemma 4.10(i), a generic linear subspace of dimension $\delta^a(\mathrm{C})$ is transverse to every irreducible component of $\mathbb{W}$. $\square$

Example 4.12. Let C be the banana curve. Then $\mathrm{Def}(\mathrm{C})$ is 2-dimensional, since C has two nodes. We have $\delta^a(\mathrm{C}) = \delta(\mathrm{C}) + 1 - \gamma(\mathrm{C}) = 2 + 1 - 2 = 1$, and indeed the relative fine compactified Jacobian of a general 1-parameter family containing a banana curve is smooth – indeed, it is the family itself.

Remark 4.13. The regularity criterion in Theorem 4.11 is sharp (in other words, the only if implication is also true) if the following Conjecture is true:

Conjecture 4.14. *Let $\overline{J}_{\mathrm{C}}$ be a fine compactified Jacobian of a (reduced and projective) connected curve C with planar singularities and let $\mathrm{C}^{\underline{\nu}}$ be a connected seminormal partial normalization of C that is maximal with these properties (or, even more generally, any connected partial normalization of C). Then there exists a sheaf $\mathcal{I} \in \overline{J}_{\mathrm{C}}$ such that $\mathrm{C}^{\mathcal{I}} = \mathrm{C}^{\underline{\nu}}$.*

The above Conjecture is easily checked to hold if C is irreducible: in this case the unique $\mathrm{C}^{\underline{\nu}}$ as in the statement of the conjecture is the normalization $\widetilde{\mathrm{C}}$ of C and it is enough to take $\mathcal{I} = \nu_*(L)$ for a line bundle L on $\widetilde{\mathrm{C}}$ of suitable degree. Therefore, if C is irreducible we have that $\mathbb{W} = \mathbb{V}^\delta$ by Lemma 4.10(ii) and Theorem 4.11 above is sharp.

The above Conjecture holds true for nodal curves by [MV, Thm. 5.1]; in particular, Theorem 4.11 is sharp if C is a nodal curve.

Finally we compare the nonsingularity of relative fine compactified Jacobians 4.11 with that of the relative Hilbert schemes. As a consequence of the results in [Sh] the following holds:

Theorem 4.15. *Let $\pi : \mathcal{C} \to S$ be a flat and projective family of (non necessarily connected) curves, with $\mathrm{C} = \mathcal{C}_o$ having planar singularities, and let $d\sigma : T_oS \to T\mathbb{V} = T\,\mathrm{Def}^{\mathrm{loc}}(\mathrm{C})$ be the differential of the associated map to the product of the first-order deformations of the singularities of C. Then:*

1. *The regularity of $\mathcal{C}^{[d]}$ along $(\mathcal{C}^{[d]})_o = \mathrm{C}^{[d]}$ depends only on $\mathrm{Im}(d\sigma)$.*
2. *If $\mathcal{C}^{[d]}$ is regular along $\mathrm{C}^{[d]}$, then $\dim \mathrm{Im}(d\sigma) \geqslant \min(d, \delta(\mathrm{C}))$.*
3. *If $\dim \mathrm{Im}(d\sigma) \geqslant d$ and is general among such subspaces, $\mathcal{C}^{[d]}$ is regular along $\mathrm{C}^{[d]}$.*
4. *$\mathcal{C}^{[d]}$ is regular along $\mathrm{C}^{[d]}$ for all d if and only if $\mathrm{Im}(d\sigma)$ is transverse to $\mathbb{V}^\delta$. In particular, this is the case if $\mathrm{Im}(\mathrm{d}\sigma)$ is a generic subspace of $\mathbb{V}$ of dimension at least $\delta(\mathrm{C})$.*

Corollary 4.16. *Let $\pi : \mathcal{C} \to S$ be as in Theorem 4.11. If $\mathcal{C}^{[d]}$ is regular along $(\mathcal{C}^{[d]})_o = \mathrm{C}^{[d]}$ for all d then any relative fine compactified Jacobian $\overline{J}_{\mathcal{C}}$ is regular along $(\overline{J}_{\mathcal{C}})_o = \overline{J}_{\mathrm{C}}$.*

Proof. It follows by comparing Theorem 4.11 with Theorem 4.15 and using that $\mathbb{V}^\delta \subseteq \mathbb{W}$. □

The implication in the above Corollary can be reversed if C is irreducible because in this case $\mathbb{W} = \mathbb{V}^\delta$ by Lemma 4.10(ii); we expect that this is no longer true whenever C is not irreducible, see Remark 4.13.

5. Support theorems

In this section we establish Theorems 1.8 and 1.16, which are the main results of this paper. We conduct two reductions: first to the case of a versal family, and second to the case of nodal curves, where the desired results were essentially established in §3. The second reduction is by the method of higher discriminants, which we now review. In this section we work over an algebraically closed field k. In particular regularity and smoothness are equivalent notions for a k-variety.

5.1. **Higher discriminants.** Higher discriminants [MS] give a-priori bound on supports which may appear in the direct image of the constant sheaf by a proper map.

Definition 5.1. Let $f : X \to Y$ be a proper map between nonsingular varieties. Let $\Delta^i f$ be the locus of $y \in Y$ such that there is no $(i-1)$ dimensional subspace of $T_y Y$ transverse to $df_x(T_x X)$ for every $x \in f^{-1}(y)$.

Theorem 5.2. [MS2] *Let $f : X \to Y$ be a proper map between nonsingular varieties.*[14] *Assume $\operatorname{codim} \Delta^i \geqslant i$ for all i, which is automatic in characteristic zero.*

If the support of a summand of $Rf_* \overline{\mathbb{Q}}_\ell$ has codimension i in Y, then it is a component of $\Delta^i f$.

By definition, the higher discriminants are determined by conditions about the smoothness of fibres over positive dimensional slices. Such criteria for Jacobians of irreducible curves were established in [FGvS, MRV2], and in §4 for reducible curves.

The fact that families of Jacobians of reducible curves can be smoothed over a base of lower dimension than the Hilbert schemes has the following consequence for the higher discriminants:

Proposition 5.3. *Let $\pi : \mathcal{C} \to B$ be an H-smooth projective family of reduced locally planar connected curves. Let $\pi^J : \overline{J}_{\mathcal{C}} \to B$ be a relative fine compactified Jacobian. Then the generic point of a d-codimensional component of Δ^d corresponds to an irreducible curve.*

Proof. Because $\pi : \mathcal{C} \to S$ is H-smooth, Corollary 4.16 ensures that $\overline{J}_{\mathcal{C}}$ is nonsingular. Choose a general point b in a d-codimensional component of Δ^d. By slicing generically through b, we can assume without loss of generality that $\dim B = d$ and hence that $b \in \Delta^{\dim B} B$. That is, by assumption there is no positive codimensional subspace of $T_b B$ transverse to the map $\pi^J : \overline{J}_{\mathcal{C}} \to B$.

By Theorem 4.15, the H-smoothness of $\mathcal{C}$ at $\mathcal{C}_b$ implies that the map $T_b B \to T\operatorname{Def}^{\mathrm{loc}}(\mathcal{C}_b)$ is transverse to the subspace $\mathbb{V}^\delta \subset T\operatorname{Def}^{\mathrm{loc}}(\mathcal{C}_b)$. If $\mathcal{C}_b$ were reducible, then any general codimension one subspace $\beta \subset T_b B$ would be transverse to $\mathbb{W}$, since $\mathbb{W}$ is a finite union of linear subspaces containing $\mathbb{V}$, but each of strictly larger dimension as per Lemma 4.10 (ii). By Theorem 4.11, β would be transverse to π^J, which is a contradiction. We conclude that $\mathcal{C}_b$ cannot be reducible. ☐

The corresponding result for the Hilbert scheme, established in [Sh], gives:

Proposition 5.4. *Let C be a projective reduced locally planar curve, let $\pi : \mathcal{C} \to \operatorname{Def}(\mathrm{C})$ be a versal family, and let $\pi^{[n]} : \mathcal{C}^{[n]} \to \operatorname{Def}(\mathrm{C})$ be the family of relative Hilbert schemes. Then,*

$$\Delta^i(\pi^{[n]}) \subseteq \operatorname{Def}(\mathrm{C})^{\delta \geqslant i} = \{s \in \operatorname{Def}(\mathrm{C}) \mid \delta(\mathcal{C}_{\overline{s}}) \geqslant i\}. \tag{35}$$

5.2. **Support theorem for fine compactified Jacobians.** Here we prove Theorem 1.8, which we restate for convenience:

Theorem 5.5. *Let $\pi : \mathcal{C} \to B$ be H-smooth. Then no summand of $R\pi^J_* \overline{\mathbb{Q}}_\ell$ has positive codimensional support. Thus, ${}^p R^i \pi^J_* \overline{\mathbb{Q}}_\ell \cong IC(\bigwedge^i R^1 \pi_{sm*} \overline{\mathbb{Q}}_\ell)$, and the stalk at $[\mathrm{C}]$ of ${}^p R^i \pi^J_* \overline{\mathbb{Q}}_\ell$ does not depend on the choice of the H-smooth family $\mathcal{C}$. Over a field of characteristic zero, $\overline{\mathbb{Q}}_\ell$ can be replaced by $\mathbb{Q}$.*

Proof. For clarity of exposition we split the proof of 5.5 in several steps. The constant sheaf $\overline{\mathbb{Q}}_\ell$ can be safely replaced by $\mathbb{Q}$ in all arguments when the base field has characteristic zero.

Lemma 5.6. *Let $\pi : \mathcal{C} \to B$ be H-smooth. Then no summand of $R\pi^J_* \overline{\mathbb{Q}}_\ell$ has positive codimensional support. Thus, ${}^p R^i \pi^J_* \overline{\mathbb{Q}}_\ell \cong IC(\bigwedge^i R^1 \pi_{sm*} \overline{\mathbb{Q}}_\ell)$,*

[14] In fact in [MS2] the result is stated for maps between arbitrary varieties with $\overline{\mathbb{Q}}_\ell$ replaced by IC_X and transversality suitably reinterpreted.

Proof. Theorem 5.2 says that the codimension i supports are contained in the i-th higher discriminant $\Delta^i(\pi^J)$. Proposition 5.3 that the only codimension i components of $\Delta^i(\pi^J)$ correspond to the equigeneric loci $\delta = i$ with irreducible generic fibre. Hence, the result holds if it holds for irreducible curves, and this is proved in [MS]. □

We need the following general fact

Lemma 5.7. *Let $f : X \to Y$ be a projective map between nonsingular varieties over an algebraically closed field k. Assume there is an open dense subset $Y^o \subset Y$ such that $f^o : f^{-1}(Y^o) \to Y^o$ is smooth and $Rf_*\overline{\mathbb{Q}}_\ell = \bigoplus IC(R^i f^o_*\overline{\mathbb{Q}}_\ell)[-i]$. Assume $D \hookrightarrow Y$ is a Cartier divisor, with $D \cap Y^o$ dense in D and $f^{-1}(D)$ is nonsingular, and set $f_D : f^{-1}(D) \to D$. Then*

$$ {}^pR^i f_{D*}\overline{\mathbb{Q}}_\ell = IC(R^i f^o_*\overline{\mathbb{Q}}_\ell)_{|D}. $$

Proof. By the decomposition theorem and proper base change

$$ \bigoplus IC(R^i f^o_*\overline{\mathbb{Q}}_\ell)_{|D}[-i] = \left(Rf_*\overline{\mathbb{Q}}_\ell\right)_D = Rf_{D*}\overline{\mathbb{Q}}_\ell = \bigoplus {}^pR^i f_{D*}\overline{\mathbb{Q}}_\ell[-i], $$

so it is enough to prove that, for every i, $IC(R^i f^o_*\overline{\mathbb{Q}}_\ell)_{|D}$ is perverse. This follows from [BBD], Corollaire 4.1.12, (see [MS2], §6 for a similar argument). □

Lemma 5.8. *Hypotheses as above, let $B' \hookrightarrow B$ be a Cartier divisor through $[\mathrm{C}]$, such that $\pi' : \mathcal{C} \times_B B' \to B'$ is H-smooth. Then*

$$ IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_\ell)_{|B'} = IC(\bigwedge^i R^1\pi'_{sm*}\overline{\mathbb{Q}}_\ell). $$

If $char k = 0$ the same holds with $\overline{\mathbb{Q}}_\ell$ replaced by $\mathbb{Q}$.

Proof. The H-smoothness of $\mathcal{C}' := \mathcal{C} \times_B B' \to B'$ implies that the set of points in B' corresponding to smooth curves is open dense. Applying Lemma 5.7, we have $IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_\ell)_{|B'} = {}^pR^i(\pi')^J_*\overline{\mathbb{Q}}_\ell$ is perverse and the statement follows from Lemma 5.6. □

Lemma 5.9. *Hypotheses as above the stalk at $[\mathrm{C}]$ of ${}^pR^i\pi^J_*\overline{\mathbb{Q}}_\ell$ does not depend on the choice of the H-smooth family $\mathcal{C}$.*

Proof. We are easily reduced to prove independence for H-smooth families $\mathcal{C}_B \to B$, with B be a closed nonsingular subset through $[\mathrm{C}]$ of $\mathrm{Def}(\mathrm{C})$. The base B is an intersection of Cartier divisors in $Def(\mathrm{C})$, and the statement follows by a repeated application of Lemma 5.8. □

This completes the proof of Theorem 5.5. □

Remark 5.10. Over the versal family, one could also prove 5.6 as follows, without using Proposition 5.3. The i-th higher discriminant $\Delta^i(\pi^J)$ of π^J is contained in $\mathrm{Def}(\mathrm{C})^{\delta\geqslant i}$ by Proposition 5.4 and Corollary 4.16. The generic point of each irreducible component of $\mathrm{Def}(\mathrm{C})^{\delta\geqslant i}$ is a nodal curve by Fact 2.1, so the result is reduced to the nodal case. By the decomposition theorem [BBD], we know that the RHS is a summand of the LHS, and that both are pure of weight zero. To conclude, it is enough to show that the weight polynomials of the stalks of the two complexes coincide at every point of the deformation of a nodal curve, i.e., for any nodal curve. We calculated the RHS in in Corollary 3.15 and the LHS in Corollary 3.10; they are equal.

Remark 5.11. The argument of Lemma 5.6 is reminiscent of that in [CL].

5.3. **Macdonald formula for reducible planar curves.** Here we prove Theorem 1.16. We recall the statement for convenience:

Theorem 5.12. *Let $\mathcal{C}_S \to B_S$ be a H-smooth independently broken family of locally planar curves over an algebraically closed field, admitting relative fine compactified Jacobians $\overline{\mathcal{J}}_S \to B_S$. By definition the relative fine compactified Jacobian of a disconnected curve is empty. Let g denote the locally constant function giving the arithmetic genus of the curves being parameterized.*

Then there are isomorphisms in $D^b_c(\coprod B_S)[[q]]$:

$$
\begin{aligned}
(q\mathbb{L})^{1-g} \bigoplus_{n=0}^{\infty} q^n R\pi_*^{[n]}\overline{\mathbb{Q}}_l &\cong \mathbb{E}xp\left((q\mathbb{L})^{1-g} \cdot \frac{\bigoplus q^i \cdot IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_l)[-i]}{(1-q)(1-q\mathbb{L})} \right) \\
&\cong \mathbb{E}xp\left((q\mathbb{L})^{1-g} \cdot \frac{\bigoplus q^i \cdot {}^pR^i\pi_*^J\overline{\mathbb{Q}}_l[-i]}{(1-q)(1-q\mathbb{L})} \right).
\end{aligned}
$$

Proof. As explained in the introduction, passage to a finite field, the decomposition theorem, Cebotarev, and Theorem 1.8 (i.e. Theorem 5.5 above, in particular the fact that the stalk at $[\mathrm{C}]$ of ${}^pR^i\pi_*^J\overline{\mathbb{Q}}_\ell = IC(\bigwedge^i R^1\pi_{sm*}\overline{\mathbb{Q}}_\ell)$ does not depend on the choice of the H-smooth family $\mathcal{C}$) allow us to reduce immediately to the case where $\mathcal{C}_S \to B_S$ is versal. We are matching semisimple perverse sheaves; it suffices to do this at the generic point of the support of each summand; by Cebotarev it suffices to match Frobenius traces at stalks of all general points in each such support. Our calculation (Lemma 5.4) of higher discriminants and Fact ii show that these all correspond to nodal curves. We are therefore reduced to checking that the Frobenius traces on both sides of the above equality are the same for the stalk at a nodal curve.

Thus, let C to be a nodal curve, let S be its set of irreducible components, for $S' \subset S$ let $\mathrm{C}_{S'}$ be the union of the corresponding irreducible components, and let $\Gamma_{S'}$ be the dual graph of $\mathrm{C}_{S'}$.

We calculated the stalk of the left hand side in Proposition 3.5.[15] Observing in addition $\chi_e^2 = [e]$, and recalling that we write $Z_{\overline{C}^\nu} := \left(\sum q^n H^*((\overline{\mathrm{C}}^\nu)^{(n)}, \overline{\mathbb{Q}}_\ell)\right)$ for the cohomological Zeta function,

$$
\begin{aligned}
\sum_{n=0}^{\infty} q^n H^*(\overline{\mathrm{C}}^{[n]}, \overline{\mathbb{Q}}_\ell) &= Z_{\overline{C}^\nu} \cdot \prod_{e\in \mathrm{E}} \left(1 - q\chi_e + q^2\mathbb{L}[e]\right) \\
&= Z_{\overline{C}^\nu} \cdot \prod_{e\in \mathrm{E}} \left((1-q\chi_e)(1-q\mathbb{L}\chi_e) + q\mathbb{L}\chi_e)\right) \\
&= Z_{\overline{C}^\nu} \cdot \sum_{J\subset \mathrm{E}} \left(\prod_{e\in J}(q\mathbb{L}\chi_e)\right)\left(\prod_{e\notin J}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)
\end{aligned}
$$

On the right hand side, we have to expand the exponential. By definition

$$
\mathbb{E}xp(\mathcal{F})_{[C_S]} := \sum_{S=\coprod S_\alpha} \prod_\alpha \mathcal{F}_{[C_{S_\alpha}]}
$$

By definition the compactified Jacobian of a disconnected curve was the empty set, so the contribution of the RHS is trivial except when all the (induced) subgraphs Γ_{S_α} are connected. The statement $S = \coprod S_\alpha$ is just saying that the vertex sets of the Γ_α partition the vertices of Γ. For these condition on graphs, we introduce the ad-hoc notation "$\{\Gamma_{S_\alpha}\} \# \Gamma$".

[15] Proposition 3.5 was stated in the Grothendieck group of varieties taken up to zeta equivalence; but by the Grothendieck-Lefschetz trace formula, it is well defined to pass by taking cohomology to the Grothendieck group of $\overline{\mathbb{Q}}_\ell$-vector spaces equipped with a $\hat{\mathbb{Z}}$ action (i.e. of Frobenius).

The sheaf $\mathcal{F}$ we are exponentiating is built out of the ${}^pR^i\pi_*^J\mathbb{Q}$. The stalk $\left(\sum q^i \cdot {}^pR^i\pi_*^J\mathbb{Q}[-i]\right)_{[C_{S_\alpha}]}$ was given according to Theorem 3.13 by the formula

$$Z_{\overline{\mathrm{C}}_\alpha^\nu} \cdot \sum_{I\in\mathscr{C}(\Gamma_\alpha)} \left(\prod_{e\in I} q\mathbb{L}\chi_e\right)\left(\prod_{e\notin I}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\left(\prod_{c\in\pi_0(\Gamma_\alpha)}(1-q\cdot[c])(1-q\mathbb{L}\cdot[c])\right)$$

Because Γ_α is connected and defined over $\mathbb{F}_\pi$, the third term in the product cancels with the denominator $(1-q)(1-q\mathbb{L})$. Thus the right hand side of our desired equality – the $\mathbb{E}xp$ of the above – becomes

$$\sum_{\{\Gamma_{S_\alpha}\}\#\Gamma}\prod_\alpha\left((q\mathbb{L})^{1-g_\alpha} Z_{\overline{\mathrm{C}}_\alpha^\nu}\cdot\sum_{I\in\mathscr{C}(\Gamma_\alpha)}\left(\prod_{e\in I}q\mathbb{L}\chi_e\right)\left(\prod_{e\notin I}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\right)$$

We have $\prod_\alpha Z_{\overline{\mathrm{C}}_\alpha^\nu} = Z_{\overline{\mathrm{C}}^\nu}$. Thus we can cancel the global geometry. Our task now is to prove the equality of

$$(q\mathbb{L})^{1-g}\sum_{J\subset\mathrm{E}}\left(\prod_{e\in J}(q\mathbb{L}\chi_e)\right)\left(\prod_{e\notin J}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right) \tag{36}$$

and

$$\sum_{\{\Gamma_{S_\alpha}\}\#\Gamma}\prod_\alpha\left((q\mathbb{L})^{1-g_\alpha}\sum_{I\in\mathscr{C}(\Gamma_\alpha)}\left(\prod_{e\in I}q\mathbb{L}\chi_e\right)\left(\prod_{e\notin I}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\right) \tag{37}$$

This is the usual combinatorics relating connected and disconnected counting. Explicitly, let us analyze one term of the sum in Equation 36. Fix $J\subset\mathrm{E}$. Decompose the graph $\mathrm{E}\backslash J$ into its connected components; this gives a partition of the vertex set. Let $\Gamma_1,\ldots,\Gamma_k$ be the corresponding induced subgraphs. We partition J into the edges $\tilde{J}$ which go between these induced subgraphs, and the edges $J_\alpha := J\cap\Gamma_\alpha$ in each. So the contribution of J,

$$(q\mathbb{L})^{1-g}\left(\prod_{e\in J}(q\mathbb{L}\chi_e)\right)\left(\prod_{e\notin J}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right),$$

can be rewritten as

$$(q\mathbb{L})^{1-g}\left(\prod_{e\in\tilde{J}}q\mathbb{L}\chi_e\right)\prod_\alpha\left(\left(\prod_{e\in J_\alpha}q\mathbb{L}\chi_e\right)\left(\prod_{e\in\Gamma_\alpha\backslash J_\alpha}(1-q\chi_e)(1-q\mathbb{L}\chi_e)\right)\right)$$

As $\tilde{J}$ is the union of collections of *all* edges between certain pairs of vertices, it is preserved by the Galois action since we assumed all the components of the curve were defined over the field. Thus $\prod_{e\in\tilde{J}}\chi_e = 1$. Denoting the partial normalization $\eta:\coprod\mathrm{C}_\alpha\to\mathrm{C}$, we have the sequence

$$0\to\mathcal{O}_{\mathrm{C}}\to\eta_*\mathcal{O}_{\coprod\mathrm{C}_\alpha}\to\bigoplus_{e\in\tilde{J}}k\to 0$$

Taking Euler characteristics gives $(1-g)+|\tilde{J}| = \sum_\alpha (1-g_\alpha)$. After these replacements, the term in question becomes:

$$\prod_\alpha \left((q\mathbb{L})^{1-g_\alpha} \left(\prod_{e \in J_\alpha} q\mathbb{L}\chi_e \right) \left(\prod_{e \in \Gamma_\alpha \setminus J_\alpha} (1-q\chi_e)(1-q\mathbb{L}\chi_e) \right) \right)$$

Recall that $\mathscr{C}$ means the subsets of the edge set of a graph whose removal does not disconnect the graph. By construction, $J_\alpha \in \mathscr{C}(\Gamma_\alpha)$. This corresponds exactly to a term of Equation 37. □

Appendix A. Generalities on the CKS complex

If $\mathscr{V}$ is a unipotent local system underlying a variation of pure Hodge structures of weight w on a product of punctured polydisks $(\mathbb{D}^*)^r \subset \mathbb{D}^r$, the paper [CKS, §1], gives a model for the stalk $IC(\mathscr{V})_0$ at $0 \in \mathbb{D}^r$ of the intersection cohomology complex of $\mathscr{V}$ and its weight filtration (see also [Sai2, §3]). This model works just as well, with minimal changes, in the étale theory, and we shortly review it here, as it plays a central role in our computations. We also provide proofs of the results discussed, as they are not available in the literature.

According to our conventions 2.1.8 the intersection cohomology complex lives in degrees $[0,\dots,\dim Y-1]$. Assume Y is a regular scheme over $\mathbb{F}_\pi$, and $D=\bigcup_{j\in J} D_j$ is a normal crossing divisor. Let $y \in D$. After étale localization we may assume that Y is some Zariski neighborhood of the origin in $\mathbb{A}^n$, with coordinate functions $t_1,\dots,t_n$, and D is defined by the equation $\prod_{j\in J} t_j = 0$, with $J=\{1,\dots,k\}$. We denote $j: Y\setminus D \to Y$.

Let $\mathscr{V}$ be a locally constant sheaf (i.e. a "lisse" sheaf) on $Y\setminus D$, tamely ramified along D, pointwise pure of weight w. Let $\Psi_1,\dots,\Psi_k$ be the nearby-cycle functors, and $\Phi_1,\dots,\Phi_k$ be the vanishing cycle functors associated with the functions $t_1,\dots,t_k$, normalized so that they preserve the middle perversity t-structure, i.e. for P a perverse sheaf, we ask that $\Psi_i(P)$ and $\Phi_j(P)$ are perverse (this is possible, see [Ill, Cor. 4.5, Cor. 4.6]).

Given $a \leqslant k$ we set $E^{(a)} := \bigcap_{j=1}^a D_j$. For $I \subseteq \{1,\dots,a\}$ we define functors

$$\Psi_I^{(a)} : D_c^b(Y) \mapsto D_c^b(E^{(a)})$$

by:

$$\Psi_I^{(a)} = \Theta_a \circ \dots \circ \Theta_1 \tag{38}$$

where

$$\Theta_i = \Psi_i \text{ if } i \in I, \text{ and } \Theta_i = \Phi_i \text{ if } i \in \{1,\dots,a\}\setminus I.$$

More precisely, the $\Psi_I^{(a)}$'s map (mixed) complexes, constructible with respect to the natural stratification of Y defined by D, to complexes on $E^{(a)}$ whose cohomology sheaves are (mixed) local systems on the strata of the induced stratification of $E^{(a)}$.

Proposition A.1. *We have the following isomorphism for the restriction of the intersection cohomology complex to E:*

(39)
$$i_E^* IC(\mathscr{V}) \simeq \mathbf{C}^\bullet(\{N_j\},\mathscr{V}) := \{0 \to \mathscr{V}_E \to \bigoplus_{|I|=1} \operatorname{Im} N_I \to \bigoplus_{|I|=2} \operatorname{Im} N_I \to \dots \to \operatorname{Im} N_J \to 0\}$$

where the differentials are given by

$$(-1)^k N_i : \operatorname{Im} N_{i_1}\cdots N_{i_k} \to \operatorname{Im} N_i N_{i_1}\cdots N_{i_k} \text{ if } i \neq \{i_1,\cdots,i_k\}. \tag{40}$$

Proof. We have the following two general facts:

(1) The functors Ψ_f, Φ_f associated with a regular function are $t-$exact with respect to the middle perversity t-structure.

(2) Let $B = \{f = 0\}$ be a principal divisor on Z and let $U := Z \backslash B \overset{j}{\hookrightarrow} Z \overset{i}{\hookleftarrow} B$. If P is a perverse sheaf on U, then the map of perverse sheaves

$$\Psi_f(IC(P)) \overset{can}{\to} \Phi_f(IC(P))$$

is surjective. This follows from the fact that i^*K is perverse, [BBD, Cor. 4.1.12] and the exact triangle

$$\overset{+1}{\to} i^*K \to \Psi_f(K) \overset{can}{\to} \Phi_f(K) \overset{+1}{\to}.$$

We first show how things work for $k = n = 2$, the general case being only notationally more complicated. One start by representing $i^*_{D_1} IC(\mathscr{V})$ by the complex of (shifted) perverse sheaves

$$i^*_{D_1} IC(\mathscr{V}) \simeq K_1 := \{0 \to \Psi_1(IC(\mathscr{V})) \overset{can}{\to} \Phi_1(IC(\mathscr{V})) \to 0\}.$$

Then

$$i^*_{D_1 \cap D_2} IC(\mathscr{V}) = i^*_{D_2} K_1 = \mathrm{Cone}(\Psi_2(K_1) \to \Phi_2(K_1)),$$

By the $t-$exactness (1) of Ψ_2 and Φ_2, we have

$$\Psi_2(K_1) \simeq \{0 \to \Psi_2 \circ \Psi_1(IC(\mathscr{V})) \xrightarrow{\Psi_2 \circ can} \Psi_2 \circ \Phi_1(IC(\mathscr{V})) \to 0\}$$

and

$$\Phi_2(K_1) \simeq \{0 \to \Phi_2 \circ \Psi_1(IC(\mathscr{V})) \xrightarrow{\Phi_2 \circ can} \Phi_2 \circ \Phi_1(IC(\mathscr{V})) \to 0\},$$

so that $i^*_{D_1 \cap D_2} IC(\mathscr{V})$ is isomorphic to the total complex associated with the double complex:

$$\begin{tikzcd}
\Phi_2 \circ \Psi_1(IC(\mathscr{V})) \arrow[r] & \Phi_2 \circ \Phi_1(IC(\mathscr{V})) \\
\Psi_2 \circ \Psi_1(IC(\mathscr{V})) \arrow[r] \arrow[u] & \Psi_2 \circ \Phi_1(IC(\mathscr{V})) \arrow[u]
\end{tikzcd}$$

this is now a complex of perverse sheaves on $E^{(2)} = D_1 \cap D_2$. We iterate this procedure: at the a-th step one has a complex of perverse sheaves on $E^{(a)} = \cap_{i=1}^a D_i$ representing $K_a := IC(\mathscr{V})_{|E^{(a)}}$ as

$$0 \to \oplus_{|I|=a} \Psi_I^{(a)} IC(\mathscr{V}) \to \oplus_{|I|=a-1} \Psi_I^{(a)} IC(\mathscr{V}) \to \cdots \to \oplus_{|I|=0} \Psi_I^{(a)} IC(\mathscr{V}).$$

where, in force of our conventions, the sum is on the subsets $I \subseteq \{1, \ldots, a\}$. We have

$$K_{a+1} := IC(\mathscr{V})_{|E^{(a+1)}} = \mathrm{Cone}\left(\Psi_{a+1}(K_a) \to \Phi_{a+1}(K_a)\right).$$

By the t-exactness of Ψ_{a+1} and Φ_{a+1},

$$\Psi_{a+1}(K_a) = \{0 \to \Psi^{(a+1)}_{|I|=a+1} IC(\mathscr{V}) \to \oplus_{\substack{|I|=a \\ a+1 \in I}} \Psi_I^{(a+1)} IC(\mathscr{V}) \to \cdots \to \oplus_{\substack{|I|=1 \\ a+1 \in I}} \Psi_I^{(a+1)} IC(\mathscr{V}) \to 0\}$$

and, similarly

$$\Phi_{a+1}(K_a) = \{0 \to \Psi^{(a+1)}_{\substack{|I|=a \\ a+1 \notin I}} IC(\mathscr{V}) \to \oplus_{\substack{|I|=a-1 \\ a+1 \notin I}} \Psi_I^{(a+1)} IC(\mathscr{V}) \to \cdots \to \oplus_{|I|=0} \Psi_I^{(a+1)} IC(\mathscr{V})\}$$

and the map of complexes $\Psi_{a+1}(IC(\mathscr{V})_{|E^{(a)}}) \to \Phi_{a+1}(IC(\mathscr{V})_{|E^{(a)}})$ is just the obvious one $\Psi_{a+1} \to \Phi_{a+1}$ component-wise. The total cone of this map is clearly

$$\{0 \to \oplus_{|I|=a+1} \Psi^{(a+1)} IC(\mathscr{V}) \to \oplus_{|I|=a} \Psi_I^{(a+1)} IC(\mathscr{V}) \to \cdots \to \oplus_{|I|=0} \Psi_I^{(a+1)} IC(\mathscr{V}) \to 0\},$$

and we go on until the k-th step.

Let $J = \{1, \dots, k\}$. We have the mixed local system $\mathscr{V}_E := \Psi_J^{(k)}(\mathscr{V})$ on $E := \bigcap_{j \in J} D_j$, endowed with an action (the "monodromy") of the additive group $\mathbb{Z}_l(1)^J$, which is quasi-unipotent, and there are given (Tate-twisted) *commuting* nilpotent endomorphisms

$$N_j : \mathscr{V}_E \to \mathscr{V}_E \otimes \mathbb{L},$$

the *logarithms* of the unipotent part of the monodromy. For $I = \{i_1, \dots, i_l\} \subseteq J$ we set $N_I := N_{i_1} \circ \dots \circ N_{i_l}$, and we have the corresponding local system $\mathrm{Im}\, N_I \subseteq \mathscr{V}_E \otimes \mathbb{L}^{|I|}$. One can check that Ψ_I does not depend on the order in which the Ψ_i's and Φ_j's are applied, and we have isomorphisms

$$\Psi_I^{(k)}(IC(\mathscr{V})) \simeq \mathrm{Im}\, N_I,$$

under which the differentials are precisely sent to the maps

$$(-1)^k N_i : \mathrm{Im}\, N_{i_1} \cdots N_{i_k} \to \mathrm{Im}\, N_i N_{i_1} \cdots N_{i_k} \text{ if } i \neq \{i_1, \cdots, i_k\}.$$

We conclude that we have the isomorphism (39) with differentials given by formula (40) $\square$

Proposition A.2. *Let $\mathbf{M}_\bullet$ be the monodromy filtration ([Del, §1.6]) associated with the nilpotent endomorphism $N := \sum a_j N_i$, for $a_j \in \mathbb{Z}_{>0}$ on $\mathscr{V}_E$. Then the complex $\mathbf{C}^\bullet(\{N_j\}, \mathscr{V})$, endowed with the filtration $\mathbf{M}_{\bullet - w}$ is isomorphic to $i_E^* IC(\mathscr{V})$ endowed with the weight filtration.*

Proof. The complex $\mathbf{C}^\bullet(\{N_j\}, \mathscr{V})$ is naturally a sub-complex of the Koszul-type complex

$$\mathbf{B}^\bullet(\{N_j\}, \mathscr{V}) := \{0 \to \mathscr{V}_E \to \bigoplus_{|I|=1} \mathscr{V}_E \otimes \mathbb{L} \to \cdots \to \mathscr{V}_E \otimes \mathbb{L}^{2|J|} \to 0\},$$

Given a regular function f on a variety Z, and setting $\{f = 0\} \overset{i}{\hookrightarrow} Z \overset{j}{\hookleftarrow} Z \backslash \{f = 0\}$, there is the exact triangle

$$\xrightarrow{+1} i^* Rj_* K \to \Psi_f(K) \xrightarrow{N} \Psi_f(K) \otimes \mathbb{L} \xrightarrow{+1}.$$

Starting from this exact triangle, an argument similar to the one above shows that

$$\mathbf{B}^\bullet(\{N_j\}, \mathscr{V}) \simeq i_E^* Rj_* \mathscr{V}$$

The inclusion $\mathbf{C}^\bullet(\{N_j\}, \mathscr{V}) \to \mathbf{B}^\bullet(\{N_j\}, \mathscr{V})$ corresponds to the natural map $i_E^* IC(\mathscr{V}) \to i_E^* Rj_* \mathscr{V}$, which is compatible with the action of Frobenius.

By [Del, Lemme 1.9.3], the monodromy filtration $\mathbf{M}_\bullet$ ([Del, §1.6]) associated with $N := \sum a_j N_i$, for $a_j \in \mathbb{Z}_{>0}$ on $\mathscr{V}_E$ is independent of the a_j's, and coincide with the weight filtration up to shifting by the weight of $\mathscr{V}$, hence this applies also to the subcomplex $\mathbf{C}^\bullet(\{N_j\}, \mathscr{V})$. $\square$

Finally we briefly comment on the case when the field of definition of our varieties is the complex field $\mathbb{C}$ and the l-adic étale theory is replaced by the theory of Mixed Hodge modules, see [Sai1, Sai2]. If $\mathscr{V}$ is a local system on an open set U of Y, underlying a pure variation of Hodge structures of weight w, then, for every $y \in Y$ the stalks $\mathcal{H}^i(IC(\mathscr{V}))_y$ are endowed with a natural Mixed Hodge structure with weights $\leqslant w + i$. Furthermore, we have the following compatibility statement:

Proposition A.3. *Suppose X is nonsingular, and $f : X \to Y$ is a projective map, smooth on the open dense $U \subseteq Y$. Let $\mathscr{V} = R^l f_* \mathbb{Q}_{|U}$. Then, for every $y \in Y$, the Decomposition Theorem identifies $\mathcal{H}^j(IC(\mathscr{V}))_y$ with a subquotient of $\mathrm{H}^{j+l}(f^{-1}(y))$, as mixed Hodge structures.*

If the open set U is the complement of a normal crossing divisor, and the local system $\mathscr{V}$ has unipotent monodromy around the branches of $Y \backslash U$, then the stalk of the intersection cohomology with its weight filtration are given by the CKS complex discussed above.

APPENDIX B. CURVE COUNTING AND SEVERI STRATA

We record here an analogue of the theorem in [Sh] comparing multiplicities of Severi strata with counts of nodal curves.

A family of curves $\pi : \mathcal{C} \to B$ determines constructible functions $\pi_*^{[n]}\mathbf{1}$ on B:

$$\pi_*^{[n]}\mathbf{1}(b) = \chi(\mathcal{C}_b^{[n]}) = \chi((R\pi^{[n]}\mathbb{Q})_b).$$

According to [GV, PT] it is natural to introduce a different family of constructible functions which differ by a linear change of variables:

Definition B.1. For a family $\pi : \mathcal{C} \to B$ of locally planar curves, we define constructible functions $\bar{n}^i$ on B by the formula

$$\sum_{n=0}^{\infty} q^{n+1-g}\pi_*^{[n]}\mathbf{1} = \sum_{i=0}^{\infty}\left(\frac{q}{(1-q)^2}\right)^{i+1-g}\bar{n}^i$$

In [PT3] it is shown that $\overline{n}^i$ vanishes at curves of cogenus less than i, and that $\bar{n}^i = 1$ at a curve smooth away from i nodes. In fact these multiplicities can be interpreted in terms of the geometry of the higher discriminants.

Theorem B.2. [Sh, GS] *Let $\pi : \mathcal{C} \to B$ be a locally versal family of locally planar curves. Let $\mathrm{mult}_{\overline{B^i_+}}$ denote the constructible function on B whose value at $b \in B$ is the multiplicity of $\overline{B^i_+}$ at b, and $\mathrm{Eu}_{\overline{B^i_+}}$ the Euler obstruction. Then $\mathrm{mult}_{\overline{B^i_+}} = \bar{n}^i = \mathrm{Eu}_{\overline{B^i_+}}$.*

A general setup in which to deduce the result on Euler obstructions is provided by the following theorem in [MS2] on pushing forward constructible functions.

Theorem B.3. *Let $f : X \to Y$ be proper; then*

$$f_*1_X = \sum_{i,\alpha} c^{i,\alpha} Eu_{\Delta^{i,\alpha}}$$

where $c^{i,\alpha}$ are some integers and $\Delta^{i,\alpha}$ are the i codimensional components of Δ^i.

According to [MS], (remember the conventions 2.1.8 for perverse sheaves), for locally versal families of *integral* locally planar curves, we have:

$$(1-q)^2\sum_{n=0}^{\infty} q^n\pi_*^{[n]}\mathbf{1} = \sum_{j=0}^{2g} q^j(-1)^j\chi({}^pR^j\pi_*^J\mathbb{Q})$$

hence we may recover the $\overline{n}^i$ from the Jacobians by the formula

$$\sum_{j=0}^{2g} q^{j-g}(-1)^j\chi({}^pR^j\pi_*^J\mathbb{Q}) = \sum_{i=0}^{g}\left(\frac{q}{(1-q)^2}\right)^{i-g}\overline{n}^i$$

Definition B.4. Let $\pi : \mathcal{C} \to B$ be a family of locally planar curves such that a local choice $\pi^J : \mathcal{J} \to B$ of a relative fine compactified Jacobian has regular total space. Then we define constructible functions $n^i : B \to \mathbb{Z}$ by the following formula:

$$\sum_{j=0}^{2g} q^{j-g}(-1)^j\chi({}^pR^j\pi_*^J\mathbb{Q}) = \sum_{i=0}^{\infty}\left(\frac{q}{(1-q)^2}\right)^{i-g} n^i$$

Evidently n^i vanishes for $i > g$, and it is easy to see that in fact it vanishes for $i > \delta$. However much stronger vanishing is common, and we do not know the precise statement. At an integral curve, we have by definition $n^i = \overline{n}^i$. More generally the relationship is more complicated, and we will clarify it shortly. First we give an interpretation of the $\overline{n}^i$ in terms of the Euler obstructions of Severi strata. Let B^i_{irr} be the locus which parameterizes irreducible curves of cogenus i.

Theorem B.5. *Let $\pi : \mathcal{C} \to B$ be a locally versal family of locally planar curves, and $\pi^J : \mathcal{J} \to B$ a relative fine compactified Jacobian. Then $n^i = \mathrm{Eu}_{\overline{B^i_{\mathrm{irr}}}} = \mathrm{mult}_{\overline{B^i_{\mathrm{irr}}}}$.*

Proof. By theorem B.3 we reduce to checking the statement when the central fibre is a nodal curve. Let Γ denote the dual graph of this curve. Note that then $\overline{B^i_{\mathrm{irr}}}$ is the union of several codimension i linear spaces, enumerated by the different ways of contracting all but i edges and leaving a graph with a single vertex. Or, what is the same, enumerated by the number f_{i-1} of different ways of removing i edges and leaving a connected complement. Thus at the central point, we have $\mathrm{Eu}_{\overline{B^i_{\mathrm{irr}}}} = f_{i-1}$. We specialize Corollary 3.15 by setting $t = -1$, exchanging q for $-q$, and rearranging slightly, to find $n^i = f_{i-1}$. The statement regarding the multiplicity will be shown in the next Lemma and Corollary. $\square$

We return to the question of how the n_i and the $\overline{n}_i$ are related. Let $C = C_1 + \ldots + C_t$ be a decomposition into irreducible components. We pick out subcurves by multi-indices: for $\vec{\alpha} \subset \{1, \ldots, t\}$, we write $C_{\vec{\alpha}} := \bigcup_{\alpha \in \vec{\alpha}} C_\alpha$.

Lemma B.6. *Let ϵ^α for $\alpha \in \{1, \ldots, t\}$ be (commuting) variables squaring to zero, and let $\epsilon^{\vec{\alpha}} = \prod_{\alpha \in \vec{\alpha}} \epsilon^\alpha$. Then*

$$\prod_{\vec{\alpha} \subset \{1,\ldots,t\}} \left(1 + \epsilon^{\vec{\alpha}} \sum_i z^{i-g(\vec{\alpha})+1} \mathrm{Eu}_{\overline{B^i_{\mathrm{irr}}}}(C_{\vec{\alpha}}) \right) = \sum_{\vec{\alpha} \subset \{1,\ldots,t\}} \epsilon^{\vec{\alpha}} \sum_i z^{i-g(\vec{\alpha})+1} \mathrm{Eu}_{\overline{B^i}}(C_{\vec{\alpha}})$$

Proof. This is essentially a tautology. The exponent on z, on both sides, is just the Euler characteristic of the structure sheaf of the normalization of a curve in the respective locus B^i or B^i_{irr} on both sides. This is additive under union of components of a curve. The locus B^i has (not necessarily irreducible) components indexed by partitions $\vec{\alpha} = \coprod \vec{\beta}_j$; the corresponding locus is the one such that the irreducible components of the curves are named by the various $\vec{\beta}_j$. These in turn have components according as to how the cogenus is distributed; say i_j to $\vec{\beta}_j$ in any way such that $\sum (i_j + 1 - g(\beta_j)) = i + 1 - g$. Analytically locally up to a smooth factor, B^i is the union of the $\prod B^{i_j}_{\mathrm{irr}}$, and the Euler obstruction is additive under union of components. $\square$

Thus according to Theorem B.5, the relation between the n^i and the $\overline{n}^i$ is:

$$\prod_{\vec{\alpha} \subset \{1,\ldots,t\}} \left(1 + \epsilon^{\vec{\alpha}} \sum_i z^{i-g(\vec{\alpha})+1} n^i(\vec{\alpha})(C_{\vec{\alpha}}) \right) = \sum_{\vec{\alpha} \subset \{1,\ldots,t\}} \epsilon^{\vec{\alpha}} \sum_i z^{i-g(\vec{\alpha})+1} \overline{n}(C_{\vec{\alpha}}) \tag{41}$$

Observe that $n^i(C_{\vec{\alpha}})$ and $\overline{n}^i(C_{\vec{\alpha}})$ determine each other modulo invariants of curves $C_{\subsetneq \alpha}$.

Corollary B.7. *We have an equality of constructible functions* $\mathrm{Eu}_{\overline{B^i_{\mathrm{irr}}}} = \mathrm{mult}_{\overline{B^i_{\mathrm{irr}}}}$.

Proof. By Theorem B.2, the formula on the RHS of Lemma B.6 does not change if Euler obstruction is replaced by multiplicity. Since both multiplicity and Euler obstruction are additive, we may conclude the same for the LHS by induction on $\#\vec{\alpha}$. $\square$

Finally we substitute in $z = q/(1-q)^2$ and replace the $n^i, \overline{n}^i$ by their defining quantities:

Corollary B.8. *Let $C = \coprod C_\alpha$ be the decomposition into irreducible components of a locally planar curve. The Euler numbers of the Hilbert schemes and the perverse Poincaré polynomials of the fine compactified Jacobians are related by the following formula:*

$$\prod_{\vec{\alpha}\subset\{1,\dots,t\}} \left(1+\epsilon^{\vec{\alpha}}\frac{q}{(1-q)^2}\sum_{j=0}^{2g(\vec{\alpha})} q^{j-g(\vec{\alpha})}(-1)^j\chi({}^pR^j(\overline{J}(C_{\vec{\alpha}})))\right) = \sum_{\vec{\alpha}\subset\{1,\dots,t\}} \epsilon^{\vec{\alpha}}\sum_{n=0}^{\infty} q^{n+1-g(\vec{\alpha})}\chi(C_{\vec{\alpha}}^{[n]})$$

Remark B.9. The assertion of Corollary B.8 plus the arguments of Corollary B.7 and Lemma B.6 would allow Theorem B.5 to be deduced from Theorem B.2.

That is, under the hypotheses of this section, the stable pairs theory of Pandharipande and Thomas [PT] is related by the expected change of variables to the prescription of Gopakumar and Vafa, so long as the moduli space of D2-branes is a relative fine compactified Jacobian.

References

[Ab] Ahmed Abbes, *Réduction semi-stable des courbes d'après Artin, Deligne, Grothendieck, Mumford, Saito, Winters*, in Courbes semi-stables et groupe fondamental en géométrie algébrique (Birkhäuser 2000), 59–110. 24

[Ale] Valery Alexeev, *Compactified Jacobians and Torelli map*, Publ. RIMS Kyoto Univ. **40** (2004), 1241–1265. 2, 22

[AIK] Allen Altman, Anthony Iarrobino, and Steven Kleiman, *Irreducibility of the Compactified Jacobian*, in Real and complex singularities: Proceedings of the Ninth Nordic Summer School, (Sijthoff and Noordhoff, 1977), 1–12. 2

[AK] Allen Altman and Steven Kleiman, *Compactifying the Picard Scheme*, Adv. in Math. **35** (1980), 50–112. 2

[AK2] Allen Altman and Steven Kleiman, *Compactifying the Picard scheme. II*, Amer. J. Math., 101.1 (1979) 10–41. 2

[Bea] Arnaud Beauville, *Counting rational curves on K3 surfaces*, Duke Math. J. **97**.1 (1999), 99–108. 3

[Beh] Kai Behrend, *Donaldson-Thomas invariants via microlocal geometry*, Annals of Math. **170** (2009), 1307–1338. 4

[BBD] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, *Faisceaux pervers*, Astèrisque **100** (1982), 5–171. 5, 10, 15, 16, 17, 36, 40

[BLR] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, Néron Models, Ergeb. der Math. und ihrer Grenz. **21** (Springer, 1990). 9

[dCHM] Mark de Cataldo, Tamás Hausel, and Luca Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case A_1*, Annals of Math. **173** (2012), 1329–1407. 8, 22

[CKS] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *L^2 and intersection cohomology for a polarized variation of Hodge structure*, Invent. Math. **87** (1987), 217–252. 8, 39

[CL] Pierre-Henri Chaudouard and Gérard Laumon, *Le lemme fondamental pondéré I : constructions géométriques*, Compos. Math. **146**.6 (2010), 1416–1506. 5, 36

[CDP] Wu-Yen Chuang, Duiliu-Emanuel Diaconescu, Guang Pan, *BPS states and the $P = W$ conjecture*, in Moduli spaces, LMS Lecture Note Ser. **411** (Cambridge Univ. Press, 2014), 132–150. 3, 8

[D'S] Cyril D'Souza, *Compactification of generalised Jacobians*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88**.5 (1979) 419–457. 2

[Del] Pierre Deligne, *La conjecture de Weil, II*, Publ. Math. IHES **52** (1980), 138–252. 15, 17, 41

[DH] Steven Diaz and Joe Harris, *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 433-468. 9, 10

[DHS] Duiliu-Emanuel Diaconescu, Zheng Hua, and Yan Soibelman, *HOMFLY polynomials, stable pairs and motivic Donaldson-Thomas invariants*, arXiv:1202.4651. 3

[DSV] Duiliu-Emanuel Diaconescu, Vivek Shende, and Cumrun Vafa, *Large N duality, lagrangian cycles, and algebraic knots*, Comm. Math. Phys. **319**.3 (2013), 813–863. 3

[Est] Eduardo Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. **353** (2001), 3045–3095. 2, 14, 15

[FGvS] Barbara Fantechi, Lothar Göttsche, and Duco van Straten, *Euler number of the compactified Jacobian and multiplicity of rational curves*, J. Algebraic Geom. **8**.1 (1999), 115--133. 8, 29, 31, 35

[Gie] David Gieseker, *On the moduli of vector bundles on an algebraic surface*, Annals of Math. **106**.1 (1977) 45–60. 2

[GORS] Eugene Gorsky, Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende, *Torus knots and the rational DAHA*, Duke Math. J. **163**.14 (2014), 2709–2794. 3, 8

[GS] Lothar Göttsche and Vivek Shende, *Refined curve counting on complex surfaces*, Geom. Topol. **18**.4 (2014), 2245–2307. 3, 4, 42

[GS2] Lothar Göttsche and Vivek Shende, *The chi-y genera of relative Hilbert schemes for linear systems on Abelian and K3 surfaces*, arXiv:1307.4316. 3

[GLS] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, Introduction to Singularities and Deformations, (Springer, 2007). 9, 10, 11, 12

[GV] Rajesh Gopakumar and Cumrun Vafa, *M-theory and Topological strings I & II*, hep-th/9809187 & hep-th/9812187 . 3, 8, 42

[Har] Robin Hartshorne, *Generalized Divisors on Gorenstein Schemes*, K-Theory **8** (1994), 287–339. 21

[HST] Shinobu Hosono, Masa-Hiko Saito, and Atsushi Takahashi, *Relative Lefschetz action and BPS state counting*, IMRN (2001) no. **15**, 783–816. 8

[Ill] Luc Illusie, *Autour du théorème de monodromie locale*, in Périodes p-adiques, Astérisque **223** (1994), 9–57. 39

[KKV] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa, *M-theory, topological strings, and spinning black holes*, Adv. Theor. Math. Phys. **3** (1999), 1445–1537. 8

[KS] Steven Kleiman and Vivek Shende, *On the Göttsche threshold*, in A celebration of algebraic geometry, Clay Math. Proc 18 (2013), 429–449. 3

[KST] Martijn Kool, Vivek Shende, and Richard Thomas, *A short proof of the Göttsche conjecture*, Geom. Topol. **15** (2011), 397–406. 3

[KT] Martijn Kool and Richard Thomas, *Reduced classes and curve counting on surfaces I: theory*, Algebraic Geometry **1**.3 (2014), 334–383 3

[KW] Reinhardt Kiehl and Rainer Weissauer, Weil Conjectures, Perverse Sheaves and l-adic Fourier Transform, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, Vol. 42 (Springer 2001). 15

[Liu] Qing Liu Algebraic geometry and arithmetic curves, (Oxford Univ. Press 2002). 30

[Lau] Gérard Laumon, *Transformation de Fourier, constantes d'equations fonctionelles et conjecture de Weil*, Pub. Math. IHES **65** (1987), 131–210. 16, 17

[Lau2] Gérard Laumon, *Fibres de Springer et jacobiennes compactifiées*, in Algebraic geometry and number theory, Progr. Math. 253 (Birkhäuser, 2006), 515–563,. 9, 10, 12

[Mac] Ian Macdonald, *The Poincare Polynomial of a Symmetric Product*, Math. Proc. of the Cambridge Philos. Soc. **58** (1962), 563–568. 28

[MV] Margarida Melo and Filippo Viviani, *Fine compactified Jacobians*, Math. Nach. **285** (2012), no. 8-9, 997–1031. 2, 22, 23, 34

[MRV1] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fine compactified Jacobians of reduced curves*, arXiv:1406.2299. 2, 5, 13, 14, 15

[MRV2] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fourier-Mukai and autoduality for compactified Jacobians I*, arxiv:1207.7233. 8, 9, 10, 11, 14, 35

[MRV3] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fourier-Mukai and autoduality for compactified Jacobians II*, arXiv:1308.0564. 2, 14

[MS] Luca Migliorini and Vivek Shende, *A support theorem for Hilbert schemes of planar curves*, J. Eur. Math. Soc. (JEMS) **15**.6 (2013), 2353–2367. 3, 5, 6, 22, 35, 36, 42

[MS2] Luca Migliorini and Vivek Shende, *Higher discriminants and the topology of algebraic maps*, arxiv:1307.4059. 5, 8, 35, 36, 42

[Mau] Davesh Maulik, *Stable pairs and the HOMFLY polynomial*, arxiv:1210.6323. 3

[MNOP] Davesh Maulik, Nikita Nekrasov, Andrei Okounkov, and Rahul Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I.*, Compos. Math. **142**.5 (2006), 1263–1285. 4

[MY] Davesh Maulik and Zhiwei Yun, *Macdonald formula for curves with planar singularities*, J. Reine Angew. Math. **694** (2014), 27–48. 3, 5, 6, 10, 11, 22

[Mil] James Milne, Étale Cohomology, Princeton Mathematical Series **33** (Princeton University Press, 1980). 18

[Ngô] Bâu Cháu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. IHES **111** (2010), 1-169. 5, 22

[OY] Alexei Oblomkov and Zhiwei Yun, *Geometric representations of graded and rational Cherednik algebras*, arxiv:1407.5685. 8

[ObS] Alexei Oblomkov and Vivek Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Math. J. **161** (2012), no. 7, 1277–1303. 3

[ORS] Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link*, arxiv:1201.2115. 3

[OS] Tadao Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90. 2, 22

[PT] Rahul Pandharipande and Richard Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178**.2 (2009), 407–447. 3, 42, 44

[PT3] Rahul Pandharipande and Richard Thomas, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. **23** (2010), 267–297. 3, 42

[Ran] Ziv Ran, *A note on Hilbert schemes of nodal curves*, J. Algebra **292**.2 (2005), 429–446. 7, 21

[Ren] Jorgen Rennemo, *Homology of Hilbert schemes of a locally planar curve*, arxiv:1308.4104. 5

[Sai1] Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24**.6 (1989), 849–995. 41

[Sai2] Morihiko Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26**.2 (1990), 221–333. 39, 41

[Ser] Edoardo Sernesi, Deformations of algebraic schemes, Grundlehren der Mathematischen Wissenschaften **334** (Springer-Verlag, 2006). 10, 11

[Ses] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96** (1982). 2

[Sh] Vivek Shende, *Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation*, Comp. Math. **148**.2 (2012) 531–547. 3, 4, 5, 8, 29, 34, 35, 42

[Sim] Carlos Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994) 47–129. 2

[Tes] Bernard Teissier, *Résolution simultanée - I. Famille de courbes*, in Séminaire sur les singularités des surfaces, Springer LNM **777** (Springer, 1980). 10, 12

[W] Thorsten Warmt, *Gorenstein-Dualität und topologische Invarianten von Singularitäten*, Dissertation, Mainz 2002. 29, 30, 31

[YZ] Shing-Tung Yau and Eric Zaslow, *BPS states, string duality, and nodal curves on K3*, Nuclear Phys. B **471**.3 (1996) 503–512. 3